# First-order methods for stochastic and finite-sum convex optimization with deterministic constraints

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#### Abstract

In this paper, we study a class of stochastic and finite-sum convex optimization problems with deterministic constraints. Existing methods typically aim to find an  $\epsilon$ -expectedly feasible stochastic optimal solution, in which the expected constraint violation and expected optimality gap are both within a prescribed tolerance  $\epsilon$ . However, in many practical applications, constraints must be nearly satisfied with certainty, rendering such solutions potentially unsuitable due to the risk of substantial violations. To address this issue, we propose stochastic first-order methods for finding an  $\epsilon$ -surely feasible stochastic optimal ( $\epsilon$ -SFSO) solution, where the constraint violation is deterministically bounded by  $\epsilon$  and the expected optimality gap is at most  $\epsilon$ . Our methods apply an accelerated stochastic gradient (ASG) scheme or a modified variance-reduced ASG scheme only once to a sequence of quadratic penalty subproblems with appropriately chosen penalty parameters. We establish first-order oracle complexity bounds for the proposed methods in computing an  $\epsilon$ -SFSO solution. As a byproduct, we also derive first-order oracle complexity results for sample average approximation method in computing an  $\epsilon$ -SFSO solution of the stochastic optimization problem using our proposed methods to solve the sample average problem.

**Keywords:** stochastic optimization, finite-sum optimization, accelerated gradient, variance reduction, single-loop scheme, quadratic penalty, sample average approximation

Mathematics Subject Classification: 90C15, 90C25, 90C30, 65K05

# 1 Introduction

In this paper, we consider constrained stochastic convex optimization problems of the form

$$F^* = \min_{x} \{F(x) := \mathbb{E}[f(x,\xi)] + \psi(x)\}$$
  
s.t.  $c(x) \le 0,$  (1)

where  $\xi$  is a random variable with sample space  $\Xi$ ,  $\tilde{f}(\cdot,\xi)$  is a continuously differentiable convex function on  $\mathbb{R}^n$  for each  $\xi \in \Xi$ ,  $\psi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$  is a proper closed convex function with an exactly evaluable proximal operator and domain denoted by  $\mathcal{X}$ , and  $c = (c_1, \ldots, c_m)$  is a Lipschitz smooth *deterministic* mapping, with each component  $c_i$  being convex.<sup>1</sup>

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<sup>&</sup>lt;sup>1</sup>For simplicity, we focus on problem (1) with convex inequality constraints only. However, our algorithms and results can be directly extended to problems that include both affine equality constraints and convex inequality constraints.

Problem (1) arises in a variety of important areas, including energy systems [29], healthcare [34], image processing [24], machine learning [9, 15], network optimization [2], optimal control [3], PDE-constrained optimization [27], resource allocation [11], and transportation [25]. More applications can be found, for example, in [6, 8, 14, 22] and references therein.

Numerous stochastic gradient methods have been developed to solve special cases of problem (1) where c = 0,  $\mathbb{E}[\tilde{f}(\cdot,\xi)]$  is Lipschitz smooth, and  $\mathcal{X}$  is compact. Notably, when  $\Xi$  is an infinite set, stochastic gradient methods (e.g., [18, 26]) achieve a first-order oracle (FO) complexity of  $\mathcal{O}(\epsilon^{-2})$ —measured by the number of gradient evaluations of  $\tilde{f}(\cdot,\xi)$ —for finding an  $\epsilon$ -stochastic optimal solution, i.e., a point x satisfying  $\mathbb{E}[F(x) - F^*] \leq \epsilon$ . On the other hand, when  $\Xi$  is a finite set, several stochastic gradient methods (e.g., [10, 28, 30, 36]) can be applied to find an  $\epsilon$ -stochastic optimal solution with improved FO complexity of  $\tilde{\mathcal{O}}(\epsilon^{-1})$ ,<sup>2</sup> where variance reduction techniques are utilized. More recently, optimal incremental gradient methods (e.g., [20]) and accelerated variance-reduced stochastic methods (e.g., [1, 19]) have been developed to achieve (nearly) optimal complexity of  $\tilde{\mathcal{O}}(\epsilon^{-1/2})$  or  $\mathcal{O}(\epsilon^{-1/2})$  for computing an  $\epsilon$ -stochastic optimal solution.

Stochastic gradient methods have also been developed for solving special instances of problem (1) with  $c \neq 0$  and  $\mathbb{E}[\tilde{f}(\cdot,\xi)]$  being Lipschitz smooth. Specifically, when  $\Xi$  is an infinite set, a stochastic gradient method is proposed in [18, Section 3.2] for solving (1) where  $\psi$  is the indicator function of a simple convex compact set and the constraints are linear equalities of the form Ax - b = 0. This method achieves an FO complexity of  $\mathcal{O}(\epsilon^{-2})$  for finding a stochastic solution x satisfying  $||Ax - b|| \leq \epsilon$  and  $F(x) - F^* \leq \epsilon$  with high probability. In addition, the constraint extrapolation method [5] can be applied to (1) to find a stochastic solution x satisfying

$$\mathbb{E}[\|[c(x)]_+\|] \le \epsilon, \quad \mathbb{E}[F(x) - F^*] \le \epsilon, \tag{2}$$

with an operation complexity of  $\mathcal{O}(\epsilon^{-2})$ . This complexity is measured by the number of convex subproblems solved, each involving simple quadratic constraints and an objective that is the sum of  $\psi$  and a simple quadratic function (where "simple" means that the Hessian is a multiple of the identity matrix). Furthermore, when  $\Xi$  is a finite set and  $\psi$  is the indicator function of a simple convex compact set, the level-set method [21] can also be applied to problem (1) to find a solution x satisfying (2), achieving an FO complexity of  $\widetilde{\mathcal{O}}(\epsilon^{-2})$ .

In many applications such as energy systems [29], machine learning [9, 15], resource allocation [11], and transportation [25], all or some of the constraints in problem (1) are hard constraints that represent imperative requirements. Consequently, any desirable approximate solution must (nearly) satisfy these constraints. In this paper, we propose stochastic first-order methods with complexity guarantees for finding an  $\epsilon$ -surely feasible stochastic optimal ( $\epsilon$ -SFSO) solution to problem (1), that is, a point x satisfying

$$\|[c(x)]_+\| \le \epsilon, \quad \mathbb{E}[|F(x) - F^*|] \le \epsilon, \tag{3}$$

by considering two separate cases for the sample space  $\Xi$ : infinite or finite.

Specifically, when  $\Xi$  is an infinite sample space, we propose a *single-loop* stochastic first-order method (Algorithm 1) to solve problem (1) by applying an accelerated stochastic gradient scheme [18] only once to a sequence of quadratic penalty problems

$$\min_{x} \left\{ \mathbb{E}[\tilde{f}(x,\xi)] + \psi(x) + \frac{\rho_k}{2} \| [c(x)]_+ \|^2 \right\}$$

for appropriately chosen penalty parameters  $\{\rho_k\} \subset (0, \infty)$ . Under suitable assumptions, we show that this method achieves a (nearly) optimal FO complexity of  $\mathcal{O}(\epsilon^{-2})$  or  $\widetilde{\mathcal{O}}(\epsilon^{-2})$  for finding an  $\epsilon$ -SFSO solution x satisfying (3), depending on whether constant or dynamic penalty parameters  $\{\rho_k\}$  are used.

 $<sup>^{2}\</sup>widetilde{\mathcal{O}}(\cdot)$  represents  $\mathcal{O}(\cdot)$  with logarithmic factors hidden.

When  $\Xi$  is a finite sample space with  $\Xi = \{1, 2, ..., s\}$ , we assume  $\xi$  follows a uniform distribution for simplicity. In this case, problem (1) reduces to a constrained finite-sum optimization problem:

$$F^* = \min_{x} \left\{ F(x) := \frac{1}{s} \sum_{i=1}^{s} f_i(x) + \psi(x) \right\}$$
  
s.t.  $c(x) \le 0,$  (4)

where  $f_i(x) := \tilde{f}(x, i)$ . To solve problem (4), we propose a stochastic first-order method (Algorithm 2) that applies a modified variance-reduced accelerated gradient scheme *only once* to a sequence of associated quadratic penalty subproblems:

$$\min_{x} \left\{ \frac{1}{s} \sum_{i=1}^{s} f_i(x) + \psi(x) + \frac{\rho_k}{2} \| [c(x)]_+ \|^2 \right\}$$
(5)

for a suitably chosen sequence of penalty parameters  $\rho_k \subset [0, \infty)$ . Our modified scheme resembles a single loop of [19, Algorithm 1], with a key distinction: the smooth components of (5), namely  $s^{-1} \sum_{i=1}^{s} f_i(x)$  and  $\rho_k ||[c(x)]_+||^2/2$ , are treated separately. Specifically, the gradient of  $s^{-1} \sum_{i=1}^{s} f_i(x)$ is approximated using a variance-reduced stochastic estimator, while the gradient of  $\rho_k ||[c(x)]_+||^2/2$  is computed exactly (see Section 3 for details). We show that under suitable assumptions and  $m = \mathcal{O}(1)$ , the proposed method achieves an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$  for finding an  $\epsilon$ -SFSO solution x satisfying (3), and an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$  or  $\widetilde{\mathcal{O}}(s \log s + \sqrt{s}\epsilon^{-1})$  for finding an  $\epsilon$ -expectedly feasible stochastic optimal ( $\epsilon$ -EFSO) solution, i.e., a point x satisfying

$$\mathbb{E}[\|[c(x)]_+\|] \le \epsilon, \quad \mathbb{E}[|F(x) - F^*|] \le \epsilon, \tag{6}$$

depending on whether constant or dynamic penalty parameters  $\rho_k$  are used. Comparing (3) with (6), it is evident that an  $\epsilon$ -SFSO solution is generally stronger than an  $\epsilon$ -EFSO solution.

As a byproduct, we analyze the FO complexity of sample average approximation (SAA) method (e.g., see [4, 16, 31, 32, 33]) for computing an  $\epsilon$ -SFSO or  $\epsilon$ -EFSO solution of problem (1) with an infinite sample space  $\Xi$ . Specifically, we establish that the SAA method, when combined with a proposed stochastic first-order method, achieves an FO complexity of  $\mathcal{O}(\epsilon^{-2})$  for computing an  $\epsilon$ -EFSO solution of (1). Moreover, we show that the FO complexity for obtaining an  $\epsilon$ -SFSO solution of (1) via SAA is either  $\mathcal{O}(\epsilon^{-3})$  or  $\mathcal{O}(\epsilon^{-5/2})$ , depending on whether (nearly) optimal deterministic first-order methods or a proposed stochastic first-order method (Algorithm 2) is used to solve the sample average problem.

The main contributions of this paper are summarized as follows.

- We propose a single-loop stochastic first-order method (Algorithm 1) for solving problem (1) with an infinite sample space. This method finds an  $\epsilon$ -EFSO solution—generally stronger than the commonly studied  $\epsilon$ -SFSO solution—with (nearly) optimal FO complexity of  $\mathcal{O}(\epsilon^{-2})$  or  $\widetilde{\mathcal{O}}(\epsilon^{-2})$ .
- We propose a stochastic first-order method (Algorithm 2) for solving problem (1) with a finite sample space. It achieves an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$  for finding an  $\epsilon$ -SFSO solution, and  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$  or  $\widetilde{\mathcal{O}}(s \log s + \sqrt{s}\epsilon^{-1})$  for finding an  $\epsilon$ -EFSO solution, assuming  $m = \mathcal{O}(1)$ .
- We establish FO complexity results for SAA method in computing an  $\epsilon$ -SFSO or  $\epsilon$ -EFSO solution of problem (1) with an infinite sample space, when either (nearly) optimal deterministic first-order methods or a proposed stochastic first-order method is used to solve the sample average problem.

To the best of our knowledge, this work is the first to study stochastic first-order methods for computing an  $\epsilon$ -EFSO solution of problem (1) with provable complexity guarantees. Moreover, in the

special case where s = 1, Algorithm 2 reduces to a single-loop first-order penalty method for solving deterministic constrained convex optimization problems, and achieves (nearly) optimal FO complexity of  $\mathcal{O}(\epsilon^{-1})$  or  $\widetilde{\mathcal{O}}(\epsilon^{-1})$ , depending on whether constant or dynamic penalty parameters are used.

The rest of the paper is organized as follows. Subsection 1.1 introduces the notation and assumptions used throughout the paper. Sections 2 and 3 present our proposed stochastic first-order methods for solving problem (1) under infinite and finite sample spaces, respectively, along with their convergence guarantees. Section 4 establishes complexity results for sample average approximation method applied to problem (1) with an infinite sample space. Numerical results are presented in Section 5, and the proofs of the main results are provided in Section 6. Finally, concluding remarks are given in Section 7.

#### **1.1** Notation and assumptions

The following notation will be used throughout this paper. Let  $\mathbb{R}^n$  denote the Euclidean space of dimension n. We use  $\langle \cdot, \cdot \rangle$  to denote the standard inner product, and  $\|\cdot\|$  to denote Euclidean norm. The notation  $[\cdot]_+$  denotes the projection operator onto the nonnegative orthant.

A mapping  $\phi$  is said to be  $L_{\phi}$ -Lipschitz continuous on a set  $\Omega$  if  $\|\phi(x) - \phi(x')\| \leq L_{\phi} \|x - x'\|$  for all  $x, x' \in \Omega$ . In addition, it is said to be  $L_{\nabla \phi}$ -smooth on  $\Omega$  if  $\|\nabla \phi(x) - \nabla \phi(x')\| \leq L_{\nabla \phi} \|x - x'\|$  for all  $x, x' \in \Omega$ , where  $\nabla \phi$  denotes the transpose of the Jacobian of  $\phi$ .

Throughout this paper, we make the following assumptions for problem (1).

- **Assumption 1.** (i) The set  $\mathcal{X}$  (i.e., the domain of  $\psi$ ) is bounded and the function F has a bounded variation on  $\mathcal{X}$ , i.e., there exist  $D_{\mathcal{X}} > 0$  and  $D_F > 0$  such that  $\max_{x,y\in\mathcal{X}} ||x-y|| \leq D_{\mathcal{X}}$  and  $\max_{x\in\mathcal{X}} F(x) \min_{x\in\mathcal{X}} F(x) \leq D_F$ .
  - (ii) Each constraint function  $c_i$  is  $L_{c_i}$ -Lipschitz continuous and  $L_{\nabla c_i}$ -smooth on  $\mathcal{X}$ , and satisfies  $|c_i(x)| \leq C_i$  for all  $x \in \mathcal{X}$  and i = 1, ..., m.
- (iii) The set of optimal Lagrange multipliers of problem (1), denoted by  $\Lambda^*$ , is nonempty. Consequently,  $\Lambda := \min_{\lambda \in \Lambda^*} \|\lambda\|$  is finite.

It follows from Assumption 1(ii) and the convexity of  $c_i$ 's that  $||[c(x)]_+||^2/2$  is  $L_{\nabla c^2}$ -smooth and convex, where

$$L_{\nabla c^2} := \sum_{i=1}^{m} \left( L_{c_i}^2 + C_i L_{\nabla c_i} \right).$$
(7)

# **2** A stochastic first-order method for problem (1) with an infinite sample space

In this section, we consider problem (1) in which  $\xi$  is a random variable with an infinite sample space  $\Xi$ . Specifically, we propose a single-loop stochastic first-order method to solve this problem by applying an accelerated stochastic gradient (ASG) scheme [18] only once to a sequence of quadratic penalty problems

$$F_{\rho_k}^* = \min_x \left\{ F_{\rho_k}(x) := \mathbb{E}[\tilde{f}(x,\xi)] + \psi(x) + \frac{\rho_k}{2} \| [c(x)]_+ \|^2 \right\}$$
(8)

for suitably chosen penalty parameters  $\{\rho_k\} \subset (0, \infty)$ . For each  $k \geq 1$ , the ASG scheme generates three points,  $y_k$ ,  $z_{k+1}$ , and  $x_{k+1}$ , as follows: it first forms a convex combination of  $x_k$  and  $z_k$  to obtain  $y_k$ ; then it computes a stochastic gradient of  $f_{\rho_k}(\cdot) := \mathbb{E}[\tilde{f}(\cdot,\xi)] + \rho_k ||[c(\cdot)]_+||^2/2$  at  $y_k$  and uses it in a proximal step associated with  $\psi$  to obtain  $z_{k+1}$ ; finally, it forms another convex combination of  $x_k$  and  $z_{k+1}$  to obtain  $x_{k+1}$ . Since c is a smooth deterministic mapping, a stochastic gradient of  $f_{\rho_k}$  at  $y_k$  can be computed as  $\nabla \tilde{f}(y^k, \xi_k) + \rho_k \nabla c(y^k) [c(y^k)]_+$ , where  $\xi_k$  is randomly sampled from  $\Xi$ . The resulting method for problem (1) with an infinite sample space is presented in Algorithm 1.

Algorithm 1 A stochastic first-order method for problem (1) with an infinite sample space

**Input:**  $x_1 \in \mathcal{X}, \{\beta_k\} \subset [1, \infty), \{\gamma_k\} \subset (0, \infty), \text{ and } \{\rho_k\} \subset (0, \infty).$ 1: Set  $z_1 = x_1$ . 2: **for** k = 1, 2, ... **do** 3:  $y_k = (1 - \beta_k^{-1})x_k + \beta_k^{-1}z_k$ . 4: Sample  $\xi_k \in \Xi$  and compute  $g_k = \nabla \tilde{f}(y_k, \xi_k) + \rho_k \nabla c(y_k)[c(y_k)]_+$ . 5:  $z_{k+1} = \operatorname{prox}_{\gamma_k \psi}(z_k - \gamma_k g_k)$ . 6:  $x_{k+1} = (1 - \beta_k^{-1})x_k + \beta_k^{-1}z_{k+1}$ . 7: **end for** 

Before presenting convergence results for Algorithm 1, we introduce the following additional assumption.

**Assumption 2.** The function  $f(\cdot) := \mathbb{E}[\tilde{f}(\cdot,\xi)]$  is  $L_{\nabla f}$ -smooth on  $\mathcal{X}$ , and  $\tilde{f}$  satisfies the following conditions:

$$\mathbb{E}[\nabla \tilde{f}(x,\xi)] = \nabla f(x), \quad \mathbb{E}[\|\nabla \tilde{f}(x,\xi) - \nabla f(x)\|^2] \le \sigma^2, \quad \max_{\xi \in \Xi} \|\nabla \tilde{f}(x,\xi)\| \le G \qquad \forall x \in \mathcal{X}$$

for some constants  $\sigma > 0$  and G > 0 independent of  $x \in \mathcal{X}$ .

The next theorem presents convergence results for Algorithm 1 with constant penalty parameters  $\{\rho_k\}$ , whose proof is deferred to Subsection 6.1.

**Theorem 1** (constant penalty parameters). Suppose that Assumptions 1 and 2 hold. Let  $L_{\nabla c^2}$ ,  $D_{\mathcal{X}}$ ,  $\Lambda$ ,  $L_{\nabla f}$ ,  $\sigma$ , and G be given in (7) and Assumptions 1 and 2, respectively. Assume that  $x_K$  is generated by Algorithm 1 for some  $K \geq 2$ , with the parameters chosen as

$$\rho_k = \rho, \quad \beta_k = \frac{k+1}{2}, \quad \gamma_k = \frac{k+1}{4(L_{\nabla f} + \rho_k L_{\nabla c^2})}$$
(9)

for all  $1 \le k < K$ , where  $\rho = K^{\frac{3}{2}}$ . Then we have

$$\|[c(x_K)]_+\| \le (6GD_{\mathcal{X}})^{1/2}K^{-3/4} + (12L_{\nabla c^2})^{1/2}D_{\mathcal{X}}K^{-1} + 2\Lambda K^{-3/2} + (12L_{\nabla f})^{1/2}D_{\mathcal{X}}K^{-7/4},$$
(10)

$$\mathbb{E}[F(x_K) - F^*] \le \left(6L_{\nabla c^2}D_{\mathcal{X}}^2 + 3\sigma^2 L_{\nabla c^2}^{-1}\right)K^{-1/2} + 6L_{\nabla f}D_{\mathcal{X}}^2K^{-2},\tag{11}$$

$$\mathbb{E}[F(x_K) - F^*] \ge -\Lambda \Big( (12L_{\nabla c^2} D_{\mathcal{X}}^2 + 6\sigma^2 L_{\nabla c^2}^{-1})^{1/2} K^{-1} + 2\Lambda K^{-3/2} + (12L_{\nabla f})^{1/2} D_{\mathcal{X}} K^{-7/4} \Big).$$
(12)

**Remark 1.** One can observe from Theorem 1 that Algorithm 1 with constant penalty parameters  $\rho_k \equiv K^{3/2}$  achieves an FO complexity of  $\mathcal{O}(\epsilon^{-2})$  for finding an  $\epsilon$ -SFSO solution of problem (1) as defined in (3). This complexity matches the optimal FO complexity achieved by stochastic gradient methods for solving the unconstrained counterpart of (1), i.e., the case where c = 0. Consequently, the presence of the constraints does not affect the order of dependence of the complexity on  $\epsilon$ .

While Algorithm 1 with a constant penalty parameter achieves the optimal complexity, the prespecified penalty parameter may be overly large in practice. To enhance its practical performance, we next consider Algorithm 1 with dynamic penalty parameters  $\{\rho_k\}$  and present its convergence results, with the proof deferred to Subsection 6.1.

**Theorem 2** (dynamic penalty parameters). Suppose that Assumptions 1 and 2 hold. Let  $L_{\nabla c^2}$ ,  $D_{\mathcal{X}}$ ,  $\Lambda$ ,  $L_{\nabla f}$ ,  $\sigma$ , and G be given in (7) and Assumptions 1 and 2, respectively. Assume that  $\{x_k\}$  is generated by Algorithm 1 with the parameters chosen as

$$\rho_k = (k+4)^{\frac{3}{2}}, \quad \beta_k = \frac{k+4}{5}, \quad \gamma_k = \frac{k+4}{10(L_{\nabla f} + \rho_k L_{\nabla c^2})}.$$
(13)

Then for all  $k \geq 2$ , we have

$$\|[c(x_k)]_+\| \le (2C_2)^{1/2} k^{-3/4} + 2(C_1 L_{\nabla c^2})^{1/2} k^{-1} + 2\Lambda k^{-3/2} + 2(C_1 L_{\nabla f})^{1/2} k^{-7/4},$$
(14)  
$$\mathbb{E}[F(x_k) - F^*] \le 3\sqrt{3}\sigma^2 L_{\nabla c^2}^{-1} k^{-1/2} \log k + 2\sqrt{3}C_1 L_{\nabla c^2} k^{-1/2} + 3\sigma^2 L_{\nabla f} L_{\nabla c^2}^{-2} k^{-2} \log k + 2C_1 L_{\nabla f} k^{-2},$$
(15)

$$\mathbb{E}[F(x_k) - F^*] \ge -\Lambda \Big( \sqrt{6}\sigma L_{\nabla c^2}^{-1/2} k^{-1} (\log k)^{1/2} + 2(C_1 L_{\nabla c^2})^{1/2} k^{-1} + 2\Lambda k^{-3/2} + \sigma L_{\nabla c^2}^{-1} (6L_{\nabla f})^{1/2} k^{-7/4} (\log k)^{1/2} + 2(C_1 L_{\nabla f})^{1/2} k^{-7/4} \Big),$$
(16)

where

$$C_1 = 25D_{\mathcal{X}}^2 + \frac{10\Lambda^2}{L_{\nabla c^2}}, \quad C_2 = \frac{20GD_{\mathcal{X}}L_{\nabla f}}{\sqrt{6}L_{\nabla c^2}} + 120GD_{\mathcal{X}}.$$
 (17)

**Remark 2.** One can observe from Theorem 2 that Algorithm 1 with dynamic penalty parameters  $\rho_k = (k+4)^{3/2}$  achieves a nearly optimal FO complexity of  $\widetilde{\mathcal{O}}(\epsilon^{-2})$  for finding an  $\epsilon$ -SFSO solution of problem (1) as defined in (3). Although this complexity is slightly worse than that achieved with constant penalty parameters, Algorithm 1 with dynamic penalties generally performs better in practice, as observed in our numerical experiments.

# **3** A stochastic first-order method for problem (1) with a finite sample space

In this section, we consider problem (1) in which  $\xi$  is a uniformly distributed random variable with a finite sample space  $\Xi = \{1, 2, ..., s\}$ . In this case, problem (1) reduces to a constrained finite-sum optimization problem given in (4).

When c = 0, problem (4) reduces to an unconstrained finite-sum optimization problem

$$\min_{x} \left\{ \bar{f}(x) + \psi(x) \right\}, \quad \text{where} \quad \bar{f}(x) := \frac{1}{s} \sum_{i=1}^{s} f_i(x).$$
(18)

This problem has been extensively studied in the literature (e.g., see [12, 19, 28, 35, 38]). Notably, [19, Algorithm 1] proposes a variance-reduced accelerated gradient method for solving (18). This method can be viewed as a restarted accelerated stochastic gradient scheme consisting of two nested loops: the outer loop updates a reference point  $\tilde{x}$  and the starting points  $x_0$ ,  $z_0$  for the inner loop, while the inner loop performs a sequence of accelerated stochastic gradient steps. At each inner iteration, a stochastic variance-reduced gradient (SVRG) of  $\bar{f}$  is computed by using the full gradient  $\nabla \bar{f}(\tilde{x})$  at the reference point  $\tilde{x}$ , along with the sampled gradients  $\nabla f_i(\tilde{x})$  and  $\nabla f_i(y)$ , where y is updated at each inner iteration, and  $i \in \{1, 2, \ldots, s\}$  is drawn according to a specified probability distribution.

We propose a stochastic first-order method for solving problem (4) by applying a *modified* variancereduced accelerated gradient scheme *only once* to a sequence of quadratic penalty problems:

$$F_{\rho_k}^* = \min_x \left\{ F_{\rho_k}(x) := \bar{f}(x) + \psi(x) + \frac{\rho_k}{2} \| [c(x)]_+ \|^2 \right\}$$
(19)

for suitably chosen penalty parameters  $\{\rho_k\} \subset [0, \infty)$ . Our modified scheme closely follows one loop of [19, Algorithm 1], but with a key distinction: the two objective components  $\bar{f}(x)$  and  $\rho_k ||[c(x)]_+||^2/2$  are handled separately. Specifically, we approximate the gradient of  $\bar{f}(x)$  using a SVRG, while computing the gradient of  $\rho_k ||[c(x)]_+||^2/2$  exactly. The sum of these two components serves as the SVRG for  $\bar{f}(x) + \rho_k ||[c(x)]_+||^2/2$  in our scheme. The resulting method for problem (4) is presented in Algorithm 2.

Algorithm 2 A variance reduced stochastic first-order method for problem (4)

**Input:**  $x^0 \in \mathcal{X}, \{\alpha_k\} \subset (0,1], \{p_k\} \subset (0,1], \{\gamma_k\} \subset (0,\infty), \{T_k\}, \{\theta_k\}, \text{ a probability distribution}$  $Q = \{q_1, q_2, \dots, q_s\}, \text{ and } \{\rho_k\} \subset (0, \infty).$ 1: Set  $\tilde{x}_1 = x^0$  and  $\tilde{z}_1 = x^0$ . 2: for k = 1, 2, ... do Set  $x_0 = \tilde{x}_k$ ,  $z_0 = \tilde{z}_k$ , and  $\tilde{g}_k = \nabla \bar{f}(\tilde{x}_k)$ . 3: for  $t = 1, 2, ..., T_k$  do 4: Sample  $i_t \in \{1, 2, \ldots, s\}$  according to  $\mathcal{Q}$ . 5:  $y_t = (1 - \alpha_k - p_k)x_{t-1} + \alpha_k z_{t-1} + p_k \tilde{x}_k.$ 6:  $g_t = (\nabla f_{i_t}(y_t) - \nabla f_{i_t}(\tilde{x}_k)) / (q_{i_t}s) + \tilde{g}_k + \rho_k \nabla c(y_t) [c(y_t)]_+.$ 7:  $z_t = \arg\min_{x \in \mathcal{X}} \left\{ \gamma_k(\langle g_t, x \rangle + \psi(x)) + \|x - z_{t-1}\|^2/2 \right\}.$ 8:  $x_t = (1 - \alpha_k - p_k)x_{t-1} + \alpha_k z_t + p_k \tilde{x}_k.$ 9: 10: end for Set  $\tilde{z}_{k+1} = z_{T_k}$  and  $\tilde{x}_{k+1} = \sum_{t=1}^{T_k} (\theta_t x_t) / \sum_{i=1}^{T_k} \theta_t$ . 11:12: **end for** 

Before presenting convergence results for Algorithm 2, we introduce the following additional assumption for problem (4).

Assumption 3.  $f_i$  is  $L_i$ -smooth on  $\mathcal{X}$  for all i = 1, 2, ..., s.

In view of this assumption and the definition of  $\bar{f}$  in (18), one can observe that  $\bar{f}$  is  $L_{\nabla \bar{f}}$ -smooth, where

$$L_{\nabla \bar{f}} = \sum_{i=1}^{s} L_i / s. \tag{20}$$

The next theorem presents convergence results for Algorithm 2 with constant penalty parameters  $\{\rho_k\}$ , whose proof is deferred to Subsection 6.2.

**Theorem 3** (constant penalty parameters). Suppose that Assumptions 1 and 3 hold. Let  $k_0 = \lfloor \log_2 s \rfloor + 1$ , and  $L_{\nabla c^2}$ ,  $L_{\nabla \bar{f}}$ ,  $D_X$ ,  $D_F$ ,  $\Lambda$ , and  $L_i$ 's be given in (7), (20), and Assumptions 1 and 3, respectively. Assume that  $\tilde{x}_K$  is generated by Algorithm 2 for some  $K \ge \max\{k_0 + 1, 2(k_0 - 3)\}$ , with the parameters chosen as

$$\alpha_{k} = \begin{cases}
\frac{1}{2} & \text{if } 1 \leq k \leq k_{0}, \\
\frac{2}{k-k_{0}+4} & \text{if } k > k_{0},
\end{cases}
\quad p_{k} = \frac{1}{2}, \quad \rho_{k} = \rho, \quad \gamma_{k} = \frac{1}{3(L_{\nabla\bar{f}} + \rho_{k}L_{\nabla c^{2}})\alpha_{k}}, \\
q_{i} = \frac{L_{i}}{\sum_{j=1}^{s}L_{j}}, \quad T_{k} = \begin{cases}
2^{k-1} & \text{if } 1 \leq k \leq k_{0}, \\
2^{k_{0}-1} & \text{if } k > k_{0},
\end{cases}
\quad \theta_{t} = \begin{cases}
\frac{\gamma_{k}}{\alpha_{k}}(\alpha_{k} + p_{k}) & \text{if } 1 \leq t \leq T_{k} - 1, \\
\frac{\gamma_{k}}{\alpha_{k}} & \text{if } t = T_{k}
\end{cases}$$
(21)

for some  $\rho > 0$  specified below. Then the following statements hold.

(i) Assume that  $\rho = s^{2/3}K^{4/3}$ . Then we have

$$\|[c(\tilde{x}_K)]_+\| \le 8L_{\nabla\bar{f}}^{1/2} D_{\mathcal{X}} s^{-1/3} K^{-2/3} + 16C_4^{1/2} s^{-1/2} K^{-1} + 2\Lambda s^{-2/3} K^{-4/3} + 16C_3^{1/2} s^{-5/6} K^{-5/3},$$
(22)

$$\mathbb{E}[\|[c(\tilde{x}_K)]_+\|] \le 16C_4^{1/2}s^{-1/2}K^{-1} + 2\Lambda s^{-2/3}K^{-4/3} + 16C_3^{1/2}s^{-5/6}K^{-5/3},$$
(23)

$$\mathbb{E}[F(\tilde{x}_K) - F^*] \le 64C_4 s^{-1/3} K^{-2/3} + 64C_3 s^{-1} K^{-2}, \tag{24}$$

$$\mathbb{E}[F(\tilde{x}_K) - F^*] \ge -16\Lambda C_4^{1/2} s^{-1/2} K^{-1} - 2\Lambda^2 s^{-2/3} K^{-4/3} - 16\Lambda C_3^{1/2} s^{-5/6} K^{-5/3}, \quad (25)$$

where

$$C_3 = 2D_F + 3L_{\nabla \bar{f}} D_{\mathcal{X}}^2/2, \quad C_4 = \|[c(x^0)]_+\|^2 + 3L_{\nabla c^2} D_{\mathcal{X}}^2/2.$$
(26)

(ii) Assume that  $\rho = \sqrt{sK}$ . Then we have

$$\begin{aligned} \|[c(\tilde{x}_{K})]_{+}\| &\leq 8L_{\nabla\bar{f}}^{1/2}D_{\mathcal{X}}s^{-1/4}K^{-1/2} + \left(2\Lambda + 16C_{4}^{1/2}\right)s^{-1/2}K^{-1} + 16C_{3}^{1/2}s^{-3/4}K^{-3/2}, \\ \mathbb{E}[\|[c(\tilde{x}_{K})]_{+}\|] &\leq \left(2\Lambda + 16C_{4}^{1/2}\right)s^{-1/2}K^{-1} + 16C_{3}^{1/2}s^{-3/4}K^{-3/2}, \\ \mathbb{E}[F(\tilde{x}_{K}) - F^{*}] &\leq 64C_{4}s^{-1/2}K^{-1} + 64C_{3}s^{-1}K^{-2}, \\ \mathbb{E}[F(\tilde{x}_{K}) - F^{*}] &\geq -\Lambda\left(2\Lambda + 16C_{4}^{1/2}\right)s^{-1/2}K^{-1} - 16\Lambda C_{3}^{1/2}s^{-3/4}K^{-3/2}, \end{aligned}$$

where  $C_3$  and  $C_4$  are given in (26).

**Remark 3.** (i) The choice of  $\rho = s^{2/3}K^{4/3}$  is to ensure that Algorithm 2 achieves the best convergence rate for both  $\|[c(\tilde{x}_K)]_+\|$  and  $|\mathbb{E}[F(\tilde{x}_K) - F^*]|$  simultaneously, thereby attaining the lowest complexity for producing an  $\epsilon$ -SFSO among all possible choices of  $\rho$ . Indeed, as observed from Lemma 8 and the proof of Theorem 3 (see (81) and (88)), we have

$$|\mathbb{E}[F(\tilde{x}_K) - F^*]| = \mathcal{O}(\rho s^{-1} K^{-2}), \qquad \|[c(\tilde{x}_K)]_+\| = \mathcal{O}(\rho^{-1/2}),$$

which implies that both  $\mathbb{E}\|[c(\tilde{x}_K)]_+\|]$  and  $|\mathbb{E}[F(\tilde{x}_K)-F^*]|$  attain the best convergence rate  $\mathcal{O}(s^{-1/3}K^{-2/3})$ when  $\rho = s^{2/3}K^{4/3}$ . On the other hand, the choice of  $\rho = \sqrt{s}K$  is to ensure that Algorithm 2 achieves the best convergence rate for both  $\mathbb{E}[\|[c(\tilde{x}_K)]_+\|]$  and  $|\mathbb{E}[F(\tilde{x}_K)-F^*]|$  simultaneously, thereby attaining the lowest complexity for producing an  $\epsilon$ -EFSO among all possible choices of  $\rho$ . Indeed, as observed from Lemma 8 and the proof of Theorem 3 (see (81) and (89)), we have

$$|\mathbb{E}[F(\tilde{x}_K) - F^*]| = \mathcal{O}(\rho s^{-1} K^{-2}), \qquad \mathbb{E}[\|[c(\tilde{x}_K)]_+\|] = \mathcal{O}(\rho^{-1}),$$

which implies that both  $\mathbb{E}[||[c(\tilde{x}_K)]_+||]$  and  $|\mathbb{E}[F(\tilde{x}_K) - F^*]|$  attain the best convergence rate  $\mathcal{O}(s^{-1/2}K^{-1})$ when  $\rho = \sqrt{sK}$ .

(ii) Assume that  $m = \mathcal{O}(1)$ . Notice from (21) that  $T_k = \mathcal{O}(s)$ , and thus the number of gradient evaluations per outer iteration of Algorithm 2 with the parameters specified in Theorem 3 is  $\mathcal{O}(s)$ . Given  $k_0 = \lfloor \log_2 s \rfloor + 1$  and  $K \ge \max\{k_0 + 1, 2(k_0 - 3)\}$ , it follows from Theorem 3 that Algorithm 2 with constant penalty parameter  $\rho_k \equiv s^{2/3}K^{4/3}$  achieves an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$ for computing an  $\epsilon$ -SFSO solution and an  $\epsilon$ -EFSO solution of (4), which significantly improves upon the complexity of Algorithm 1 in terms of the dependence of  $\epsilon$ . On the other hand, Algorithm 2 with constant penalty parameter  $\rho_k \equiv \sqrt{s}K$  achieves an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-2})$  for computing an  $\epsilon$ -SFSO solution, and an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$  for finding an  $\epsilon$ -EFSO solution of (4). Therefore, the choice  $\rho_k \equiv s^{2/3}K^{4/3}$  leads to better complexity for obtaining an  $\epsilon$ -SFSO solution, while  $\rho_k \equiv \sqrt{s}K$  yields better complexity for computing an  $\epsilon$ -EFSO solution, which is a weaker notion of approximate optimality than an  $\epsilon$ -SFSO solution. In addition, the complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$ achieved by Algorithm 2 with  $\rho_k \equiv \sqrt{s}K$  for finding an  $\epsilon$ -EFSO solution is significantly lower than the complexity of  $\mathcal{O}(s\epsilon^{-1})$  or  $\widetilde{\mathcal{O}}(s\epsilon^{-1})$  achieved by (nearly) optimal first-order methods [17, 23, 37] for finding a deterministic  $\epsilon$ -optimal solution.

(iii) In the special case s = 1, Algorithm 2 with  $\rho_k \equiv K$  reduces to a single-loop first-order penalty method with a constant penalty parameter for solving deterministic constrained convex optimization problems, while achieving an optimal FO complexity of  $\mathcal{O}(\epsilon^{-1})$ .

While Algorithm 2 with constant penalty parameters achieves strong complexity guarantees, the pre-specified penalty parameters may be overly large in practice. To enhance its practical performance, we next consider Algorithm 2 with dynamic penalty parameters  $\{\rho_k\}$  and present its convergence results, with the proofs deferred to Subsection 6.2.

**Theorem 4** (dynamic penalty parameters). Suppose that Assumptions 1 and 3 hold. Let  $L_{\nabla c^2}$ ,  $L_{\nabla \bar{f}}$ ,  $D_{\mathcal{X}}$ ,  $D_F$ ,  $\Lambda$ , and  $L_i$ 's be given in (7), (20), and Assumptions 1 and 3, respectively. Assume that  $\{x_k\}$  is generated by Algorithm 2 with the parameters chosen as

$$\theta_{t} = \begin{cases} \frac{\gamma_{k}}{\alpha_{k}} (\alpha_{k} + p_{k}) & \text{if } 1 \leq t \leq T_{k} - 1, \\ \frac{\gamma_{k}}{\alpha_{k}} & \text{if } t = T_{k}, \end{cases} \qquad \alpha_{k} = \begin{cases} \frac{6}{7} & \text{if } 1 \leq k \leq k_{0}, \\ \frac{6}{k - k_{0} + 7} & \text{if } k > k_{0}, \end{cases}$$

$$p_{k} = \frac{1}{7}, \quad \gamma_{k} = \frac{1}{8(L_{\nabla\bar{f}} + \rho_{k}L_{\nabla c^{2}})\alpha_{k}}, \quad q_{i} = \frac{L_{i}}{\sum_{j=1}^{s}L_{j}}, \qquad (27)$$

and  $k_0$ ,  $T_k$ ,  $\rho_k$  specified below. Then the following statements hold.

(i) Assume that

$$k_{0} = \lfloor (4/3) \log_{2} s \rfloor + 1,$$

$$T_{k} = \begin{cases} \lceil 2^{3(k-1)/4} \rceil & \text{if } 1 \le k \le k_{0}, \\ T_{k_{0}} & \text{if } k > k_{0}, \end{cases} \quad \rho_{k} = \begin{cases} 2^{k/2} & \text{if } 1 \le k \le k_{0}, \\ 3s^{2/3}(k-k_{0}+7)^{4/3}/32 & \text{if } k > k_{0}. \end{cases}$$
(28)

Then for all  $k \ge \max\{k_0 + 1, 2(k_0 - 7)\}$ , we have

$$\|[c(\tilde{x}_k)]_+\| \le 117L_{\nabla\bar{f}}^{1/2}D_{\mathcal{X}}s^{-1/3}k^{-2/3} + 10(C_6^{1/2} + C_8^{1/2})s^{-1/3}k^{-1} + (54\Lambda + 12C_9^{1/2})s^{-2/3}k^{-4/3} + 15(C_5^{1/2} + C_7^{1/2})s^{-2/3}k^{-5/3},$$
(29)

$$\mathbb{E}[\|[c(\tilde{x}_k)]_+\|] \le 10C_6^{1/2}s^{-1/2}k^{-1} + 54\Lambda s^{-2/3}k^{-4/3} + 15C_5^{1/2}s^{-5/6}k^{-5/3},$$
(30)

$$\mathbb{E}[F(\tilde{x}_k) - F^*] \le 2C_6 s^{-1/3} k^{-2/3} + 4C_5 s^{-1} k^{-2}, \tag{31}$$

$$\mathbb{E}[F(\tilde{x}_k) - F^*] \ge -\Lambda \left(10C_6^{1/2}s^{-1/2}k^{-1} + 54\Lambda s^{-2/3}k^{-4/3} + 15C_5^{1/2}s^{-5/6}k^{-5/3}\right),$$
(32)

where

$$C_{5} = 4032L_{\nabla\bar{f}} \Big( \frac{D_{F} + \sqrt{2} \|[c(x^{0})]_{+}\|^{2}/2}{40(L_{\nabla\bar{f}} + \sqrt{2}L_{\nabla c^{2}})} + \frac{D_{\mathcal{X}}^{2}}{2} + \frac{5\Lambda^{2}}{L_{\nabla c^{2}}} \Big),$$

$$C_{6} = 3L_{\nabla\bar{f}}^{-1}L_{\nabla c^{2}}C_{5}/32, \qquad C_{7} = 8871L_{\nabla c^{2}}^{-1}L_{\nabla\bar{f}}^{2}D_{\mathcal{X}}^{2},$$

$$C_{8} = 832L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}, \qquad C_{9} = 2688L_{\nabla c^{2}}^{-1}L_{\nabla\bar{f}}^{2}D_{\mathcal{X}}^{2}.$$
(33)

(ii) Assume that

$$k_{0} = \lfloor \log_{2} s \rfloor + 1, \quad T_{k} = \begin{cases} 2^{k-1} & \text{if } 1 \leq k \leq k_{0}, \\ T_{k_{0}} & \text{if } k > k_{0}, \end{cases} \quad \rho_{k} = \begin{cases} 2^{k/2} & \text{if } 1 \leq k \leq k_{0}, \\ 3\sqrt{s}(k-k_{0}+7)/16 & \text{if } k > k_{0}. \end{cases}$$
(34)

Then for all  $k \ge \max\{k_0 + 1, 2(k_0 - 7)\}$ , we have

$$\|[c(\tilde{x}_k)]_+\| \le 60L_{\nabla\bar{f}}^{1/2} D_{\mathcal{X}} s^{-1/4} k^{-1/2} + 150\Lambda s^{-1/2} k^{-1} \log k + 7 \left(4\Lambda + C_{11}^{1/2} + C_{14}^{1/2} + C_{15}^{1/2}\right) s^{-1/4} k^{-1} + 10C_{12}^{1/2} s^{-3/4} k^{-3/2} \log k + 10 \left(C_{10}^{1/2} + C_{13}^{1/2}\right) s^{-1/2} k^{-3/2},$$
(35)

$$\mathbb{E}[\|[c(\tilde{x}_k)]_+\|] \le 150\Lambda s^{-1/2} k^{-1} \log k + (22\Lambda + 7C_{11}^{1/2}) s^{-1/2} k^{-1} + 10C_{12}^{1/2} s^{-3/4} k^{-3/2} \log k + 10C_{10}^{1/2} s^{-3/4} k^{-3/2},$$
(36)

$$\mathbb{E}[F(\tilde{x}_k) - F^*] \le 1046\Lambda^2 s^{-1/2} k^{-1} \log k + 2C_{11} s^{-1/2} k^{-1} + 4C_{12} s^{-1} k^{-2} \log k + 4C_{10} s^{-1} k^{-2},$$
(37)

$$\mathbb{E}[F(\tilde{x}_k) - F^*] \ge -\Lambda \Big( 150\Lambda s^{-1/2} k^{-1} \log k + (22\Lambda + 7C_{11}^{1/2}) s^{-1/2} k^{-1} \\ + 10C_{12}^{1/2} s^{-3/4} k^{-3/2} \log k + 10C_{10}^{1/2} s^{-3/4} k^{-3/2} \Big),$$
(38)

where

$$C_{10} = 4032L_{\nabla\bar{f}} \Big( \frac{D_F + \sqrt{2} \|[c(x^0)]_+\|^2/2}{40(L_{\nabla\bar{f}} + \sqrt{2}L_{\nabla c^2})} + \frac{D_{\mathcal{X}}^2}{2} + \frac{5\Lambda^2 k_0}{24L_{\nabla c^2}} \Big),$$

$$C_{11} = 3L_{\nabla\bar{f}}^{-1} L_{\nabla c^2} C_{10}/16, \qquad C_{12} = 2823L_{\nabla c^2}^{-1} L_{\nabla\bar{f}}\Lambda^2, \qquad C_{13} = 2840L_{\nabla c^2}^{-1} L_{\nabla\bar{f}}^2 D_{\mathcal{X}}^2, \qquad (39)$$

$$C_{14} = 533L_{\nabla\bar{f}} D_{\mathcal{X}}^2, \qquad C_{15} = 896L_{\nabla c^2}^{-1} L_{\nabla\bar{f}}^2 D_{\mathcal{X}}^2.$$

**Remark 4.** (i) For similar reasons as discussed in Remark 3(i), the choices of  $\rho_k$  in (28) and (34) are to ensure that Algorithm 2 achieves the lowest complexity in finding an  $\epsilon$ -SFSO and an  $\epsilon$ -EFSO, respectively, among all possible choices of the sequence  $\rho_k$ .

(ii) Assume that  $m = \mathcal{O}(1)$ . Using similar arguments as in Remark 3(ii), it follows from Theorem 4 that Algorithm 2 with  $(k_0, T_k, \rho_k)$  chosen as in (28) achieves an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$ for computing both an  $\epsilon$ -SFSO solution and an  $\epsilon$ -EFSO solution of problem (4). In contrast, choosing  $(k_0, T_k, \rho_k)$  as in (34) results in an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-2})$  for finding an  $\epsilon$ -SFSO solution, and  $\widetilde{\mathcal{O}}(s \log s + \sqrt{s}\epsilon^{-1})$  for finding an  $\epsilon$ -EFSO solution.

(iii) In the special case s = 1, Algorithm 2 with dynamic penalty parameters chosen in (34) becomes the first single-loop first-order penalty method with dynamic penalty parameters for solving deterministic constrained convex optimization problems, while achieving a nearly optimal FO complexity of  $\widetilde{\mathcal{O}}(\epsilon^{-1})$ .

(iv) Although the theoretical complexity bounds of Algorithm 2 with dynamic penalty parameters are comparable to or slightly worse than those with constant penalty parameters, our numerical experiments demonstrate that the dynamic-penalty variant consistently outperforms its constant-penalty counterpart in practice.

# 4 Sample average approximation method for problem (1)

In this section, we establish FO complexity results for SAA method in computing an  $\epsilon$ -SFSO or  $\epsilon$ -EFSO solution of problem (1) with an infinite sample space using (nearly) optimal deterministic first-order methods or Algorithm 2 to solve the sample average problem, and compare these results with those achieved by Algorithm 1.

Throughout this section, we impose an additional assumption on problem (1) regarding the uniformly bounded variation of  $\tilde{f}(x,\xi)$  on  $\mathcal{X} \times \Xi$ , where  $\mathcal{X} = \operatorname{dom} \psi$  and  $\Xi$  is the sample space of the random variable  $\xi$ .

Assumption 4. There exists a constant V > 0 such that  $|\tilde{f}(x,\xi) - \tilde{f}(x',\xi')| \leq V$  for all  $(x,\xi), (x',\xi') \in \mathcal{X} \times \Xi$ .

SAA method is a classical approach for solving various stochastic optimization problems by replacing the original problem with a sample-based deterministic approximation (e.g., see [4, 16, 31, 32, 33]). Specifically, let  $\xi_1, \xi_2, \ldots, \xi_s$  be *s* independent samples of the random variable  $\xi$ . The SAA method approximates problem (1) by the following sample average problem:

$$\bar{F}^* = \min_{x} \left\{ \bar{F}(x) := \frac{1}{s} \sum_{i=1}^{s} \tilde{f}(x,\xi_i) + \psi(x) \right\}$$
  
s.t.  $c(x) \le 0.$  (40)

As shown below, if the sample size s is chosen appropriately, an approximate optimal solution of (40) also serves as an approximate solution to (1). Therefore, problem (1) can be approximately solved by solving (40).

Let  $\epsilon > 0$  be given, and set the sample size to  $s = \lceil 8\pi V^2 \epsilon^{-2} \rceil$ . Let  $\tilde{x}$  be an approximate solution to problem (40) obtained by a deterministic or stochastic first-order method (to be discussed shortly), and suppose it satisfies  $\mathbb{E}[|\bar{F}(\tilde{x}) - \bar{F}^*|] \leq \epsilon/2$ . By Hoeffding's inequality [13], one has that

$$\mathbb{P}(|\bar{F}(x) - F(x)| \ge \tau) \le 2e^{-2s\tau^2/V^2} \qquad \forall x \in \mathcal{X}, \tau > 0$$

It then follows that for all  $x \in \mathcal{X}$ ,

$$\mathbb{E}[|\bar{F}(x) - F(x)|] = \int_0^\infty \mathbb{P}(|\bar{F}(x) - F(x)| \ge \tau) d\tau \le \int_0^\infty 2e^{-2s\tau^2/V^2} d\tau$$
$$= \frac{V}{\sqrt{2s}} \int_0^\infty 2e^{-u^2} du = V\sqrt{\frac{\pi}{2s}}.$$

This implies  $\mathbb{E}[|\bar{F}^* - F^*|] \leq V\sqrt{\pi/(2s)}$ , where  $F^*$  is the optimal value of problem (1). Combining these bounds and using  $s = \lceil 8\pi V^2 \epsilon^{-2} \rceil$ , we obtain that

 $\mathbb{E}[|F(\tilde{x}) - F^*|] \le \mathbb{E}[|\bar{F}(\tilde{x}) - F(\tilde{x})|] + \mathbb{E}[|\bar{F}^* - F^*|] + \mathbb{E}[|\bar{F}(\tilde{x}) - \bar{F}^*|] \le 2V\sqrt{\pi/(2s)} + \epsilon/2 \le \epsilon.$ (41)

Hence, the expected optimality gap  $\mathbb{E}[|F(\tilde{x}) - F^*|]$  is at most  $\epsilon$ .

Based on the above observation, we can use SAA method to compute an  $\epsilon$ -SFSO or  $\epsilon$ -EFSO solution of problem (1) by instead solving the sample average problem (40). In particular, we consider three different approaches for solving (40), each resulting in a distinct FO complexity for finding an  $\epsilon$ -SFSO or  $\epsilon$ -EFSO solution of (1):

• Approach 1: Apply Algorithm 2 with parameters specified in (27) and (34) to compute an  $\epsilon/2$ -EFSO solution  $\tilde{x}$  of problem (40) such that

$$\mathbb{E}[|\bar{F}(\tilde{x}) - \bar{F}^*|] \le \epsilon/2 \quad \text{and} \quad \mathbb{E}[||[c(\tilde{x})]_+||] \le \epsilon/2,$$

which is also an  $\epsilon$ -EFSO solution of problem (1) due to (41). This approach achieves an FO complexity of  $\widetilde{\mathcal{O}}(s \log s + \sqrt{s}\epsilon^{-1}) = \widetilde{\mathcal{O}}(\epsilon^{-2})$ .

• Approach 2: Apply Algorithm 2 with parameters specified in (27) and (28) to compute an  $\epsilon/2$ -SFSO solution  $\tilde{x}$  of problem (40) such that

 $\mathbb{E}[|\bar{F}(\tilde{x}) - \bar{F}^*|] \le \epsilon/2 \quad \text{and} \quad \|[c(\tilde{x})]_+\| \le \epsilon/2,$ 

which is also an  $\epsilon$ -SFSO solution of problem (1) due to (41). This approach achieves an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2}) = \mathcal{O}(\epsilon^{-5/2})$ .

• Approach 3: Apply (nearly) optimal deterministic first-order methods (e.g., [17, Algorithm 3], [23, Algorithm 2], and [37, Algorithm 1]) to compute an  $\epsilon/2$ -optimal solution  $\tilde{x}$  of problem (40) such that

$$|F(\tilde{x}) - F^*| \le \epsilon/2$$
 and  $||[c(\tilde{x})]_+|| \le \epsilon/2.$ 

which is also an  $\epsilon$ -SFSO solution of problem (1) due to (41). This approach achieve an FO complexity of  $\mathcal{O}(\epsilon^{-3})$  or  $\widetilde{\mathcal{O}}(\epsilon^{-3})$ .<sup>3</sup>

Recall from Section 2 that an  $\epsilon$ -SFSO solution of problem (1) can be computed by Algorithm 1 with parameters specified in (9) or (13), achieving an FO complexity of  $\mathcal{O}(\epsilon^{-2})$  or  $\tilde{\mathcal{O}}(\epsilon^{-2})$ . We now compare Algorithm 1 with the SAA method under the three approaches for solving the sample average problem discussed above. Although both Algorithm 1 and the SAA method using Approach 1 achieve comparable FO complexity, the former computes an  $\epsilon$ -SFSO solution, which is generally stronger than the  $\epsilon$ -EFSO solution obtained by the latter. Furthermore, while both Algorithm 1 and the SAA method using Approaches 2 and 3 compute an  $\epsilon$ -SFSO solution, Algorithm 1 achieves significantly better FO complexity.

<sup>&</sup>lt;sup>3</sup>As these methods have iteration complexity  $\mathcal{O}(\epsilon^{-1})$  or  $\widetilde{\mathcal{O}}(\epsilon^{-1})$  and require computing gradients of all  $\tilde{f}(\cdot,\xi_i)$  at each iteration, resulting in a total FO complexity of  $\mathcal{O}(\epsilon^{-3})$  or  $\widetilde{\mathcal{O}}(\epsilon^{-3})$ .

# 5 Numerical results

In this section, we conduct preliminary experiments to test the performance of our proposed methods (namely, Algorithms 1 and 2). Both algorithms are coded in Matlab and all the computations are performed on a desktop with a 2.00 GHz Intel i9-14900F 24-core processor and 32 GB of RAM.

#### 5.1 Constrained stochastic logistic regression model

In this subsection, we consider a constrained stochastic logistic regression model:

$$\min_{w,b} - \mathbb{E}\left[\frac{1+Y}{2}\log\left(\frac{1}{1+e^{-(w^{\top}X+b)}}\right) + \frac{1-Y}{2}\log\left(\frac{e^{-(w^{\top}X+b)}}{1+e^{-(w^{\top}X+b)}}\right)\right] + \lambda \|w\|_{1} + \delta_{[-1,1]^{n+1}}(w,b)$$
s.t.  $-\tilde{y}_{i}(w^{\top}\tilde{x}_{i}+b) \leq 0, \quad i = 1, 2, \dots, m,$ 
(42)

where  $\delta_{[-1,1]^{n+1}}(\cdot)$  is the indicator function of the box set  $[-1,1]^{n+1}$ ,  $X \in \mathbb{R}^n$  is a standard normal random vector,  $Y \in \{-1,1\}$  is a binary random variable, and  $\{(\tilde{x}_i, \tilde{y}_i)\}_{i=1}^m$  are *m* core samples, which are generated as follows. Specifically, we first generate a ground-truth parameter pair  $(\hat{w}, \hat{b})$  with each entry independently sampled from the uniform distribution on [-1,1]. Given a realization X = x, the conditional distribution of Y follows the logistic model:

$$P(Y = 1 | X = x) = \frac{1}{1 + e^{-(\hat{w}^{\top} x + \hat{b})}}$$

Using this model, we generate 10,000 independent samples of (X, Y), with X drawn from a standard normal distribution. Among these, we select the m samples  $\{(\tilde{x}_i, \tilde{y}_i)\}_{i=1}^m$  that are closest to the decision boundary  $\hat{w}^{\top}x + \hat{b} = 0$ , and use them as the core samples. To ensure feasibility of problem (42), we reset the labels of the core samples as  $\tilde{y}_i = \operatorname{sgn}(\hat{w}^{\top}\tilde{x}_i + \hat{b})$ , where  $\operatorname{sgn}(\cdot)$  denotes the sign function.

We generate problem (42) with  $\lambda = 0.1$ , n = 100, and m = 50 according to the procedure described above. We then apply Algorithm 1 to solve (42) under both constant and dynamic penalty parameter settings. Specifically, we initialize the algorithm at  $(w_0, b_0)$ , with each component independently drawn from the uniform distribution on [-1, 1]. The remaining parameters of Algorithm 1 are set according to (9) with K = 5,000 for the constant penalty setting, and according to (13) for the dynamic penalty setting.

We run Algorithm 1 for 5,000 iterations and evaluate both the objective value and constraint violation of problem (42) at each iteration. The objective value at a given point (w, b) is approximated by averaging over the 10,000 previously generated samples of (X, Y). The results are presented in Figure 1, which illustrates the convergence behavior of the algorithm under both constant and dynamic penalty parameter settings. As shown in the figure, the dynamic penalty setting leads to significantly faster convergence in the objective value, while the constant penalty setting yields slightly faster reduction in constraint violation.

#### 5.2 Constrained finite-sum logistic regression model

In this subsection, we consider a finite-sum logistic regression model:

$$\min_{w,b} - \frac{1}{s} \sum_{i=1}^{s} \left[ \frac{1+y_i}{2} \log \left( \frac{1}{1+e^{-(w^{\top}x_i+b)}} \right) + \frac{1-y_i}{2} \log \left( \frac{e^{-(w^{\top}x_i+b)}}{1+e^{-(w^{\top}x_i+b)}} \right) \right] + \lambda \|w\|_1 + \delta_{[-1,1]^{n+1}}(w,b)$$
s.t.  $-\tilde{y}_i(w^{\top}\tilde{x}_i+b) \le 0, \quad i=1,2,\ldots,m,$ 
(43)



Figure 1: Performance of Algorithm 1 with constant and dynamic penalty parameters.

where  $\delta_{[-1,1]^{n+1}}(\cdot)$  is the indicator function of  $[-1,1]^{n+1}$ . The dataset  $\{(x_i, y_i)\}_{i=1}^s$  consists of s = 32,561 samples with n = 123 features from LIBSVM a1a dataset [7]. The set  $\{(\tilde{x}_i, \tilde{y}_i)\}_{i=1}^m$  contains m core samples generated as follows: we first solve the unconstrained counterpart of (43) (with  $\lambda = 0$  and m = 0) to obtain a reference solution  $(\hat{w}, \hat{b})$ . We then select the m samples closest to the decision boundary  $\hat{w}^{\top}x + \hat{b} = 0$  and use them as the core samples. To ensure feasibility of problem (43), we reset the labels of the core samples as  $\tilde{y}_i = \operatorname{sgn}(\hat{w}^{\top}\tilde{x}_i + \hat{b})$ .

We generate problem (43) with  $\lambda = 0.03$ , m = 50, and s = 32,561 using the procedure described above. We then apply Algorithm 2 to solve it under both constant and dynamic penalty parameter settings. The algorithm is initialized at  $(\hat{w}_0, \hat{b}_0)$ , where each component is independently drawn from the uniform distribution on [-1, 1]. The remaining parameters for Algorithm 2 are set according to Theorem 3 with K = 500 for the constant penalty setting and Theorem 4 for the dynamic penalty setting, respectively.



Figure 2: Performance of Algorithm 2 with constant and dynamic penalty parameters.

We run Algorithm 2 for 500 iterations and evaluate the objective value and constraint violation at each iterate. The results are presented in Figure 2, which illustrates the convergence behavior of the algorithm under both constant and dynamic penalty settings. As observed, the dynamic penalty schemes generally lead to faster convergence in objective value compared to their constant counterparts, while incurring only a slightly slower convergence in constraint violation. In addition, both the constant penalty settings  $\rho_k \equiv s^{2/3} K^{4/3}$  and its dynamic counterpart given in (28) outperform the constant penalty  $\rho_k \equiv \sqrt{s}K$  and its corresponding dynamic version given in (34) in terms of constraint violation convergence, although they exhibit slower convergence in objective value.

We also compare Algorithm 2 with an optimal deterministic first-order method—the extra anchored gradient (EAG) method—for solving problem (43). EAG, proposed in [17, Algorithm 3], is an optimal first-order method designed for monotone inclusion problems. Since problem (43) can be equivalently reformulated as a monotone inclusion problem, EAG is well-suited for solving it. The parameters of Algorithm 2 are set according to Theorem 3(ii) with K = 500 for the constant penalty setting, and Theorem 4(ii) for the dynamic penalty setting. The parameters for EAG are chosen as specified in [17, Theorem 2]. Both Algorithm 2 and EAG are run for 500 iterations. The results are presented in Figure 2, which illustrates the convergence behavior of the methods as the number of gradient evaluations increases. It can be observed that Algorithm 2, under both constant and dynamic penalty settings, significantly outperforms EAG in terms of both objective value and constraint violation.



Figure 3: Comparison of Algorithm 2 with EAG

### 6 Proof of the main results

In this section, we provide a proof of our main results presented in Sections 2 and 3, which are particularly Theorems 1, 2, 3, and 4. To proceed, we establish a lemma regarding a relationship between problems (1) and (8), which will be used in our subsequent analysis.

**Lemma 1.** Consider the penalty problem associated with (1):

$$F_{\rho}^{*} = \min_{x} \left\{ F_{\rho}(x) := F(x) + \frac{\rho}{2} \| [c(x)]_{+} \|^{2} \right\},$$
(44)

where F is defined in (1), and  $\rho > 0$ . Suppose that Assumption 1(iii) holds, and let  $\Lambda$  be given in Assumption 1(iii). Then for any  $x \in \mathcal{X}$  and  $\rho > 0$ , one has

$$-\Lambda \| [c(x)]_{+} \| \le F(x) - F^{*} \le F_{\rho}(x) - F^{*}_{\rho},$$
(45)

$$\|[c(x)]_{+}\| \le \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_{\rho}(x) - F_{\rho}^{*})}{\rho}}.$$
(46)

*Proof.* By (1) and (8), one has  $F_{\rho}(x) \ge F(x)$  and  $F_{\rho}^* \le F^*$ , which imply that

$$F(x) - F^* \le F_{\rho}(x) - F_{\rho}^*.$$
 (47)

By the definition of  $\Lambda$  in Assumption 1(iii), there exists an optimal Lagrange multiplier  $\lambda^*$  for problem (1) satisfying  $\Lambda = \|\lambda^*\|$ . Hence, one has  $\lambda^* \ge 0$  and

$$F^* = \min_{z} \{F(z) + \langle \lambda^*, c(z) \rangle \} \le F(x) + \langle \lambda^*, c(x) \rangle.$$

Using these and  $\Lambda = \|\lambda^*\|$ , we obtain that

$$F(x) - F^* \ge -\langle \lambda^*, c(x) \rangle \ge -\langle \lambda^*, [c(x)]_+ \rangle \ge -\Lambda \| [c(x)]_+ \|_{\mathcal{H}}$$

which together with (47) leads to (45). Also, by this relation,  $F_{\rho}(x) = F(x) + \rho ||[c(x)]_+||^2/2$ , and  $F_{\rho}^* \leq F^*$ , one has

$$-\Lambda \| [c(x)]_+ \| \le F(x) - F^* = F_{\rho}(x) - \frac{\rho}{2} \| [c(x)]_+ \|^2 - F^* \le F_{\rho}(x) - \frac{\rho}{2} \| [c(x)]_+ \|^2 - F_{\rho}^* + \frac{\rho}{2} \| [c(x)]_+ \|^2 + \frac{\rho}{$$

and hence

$$\frac{\rho}{2} \| [c(x)]_+ \|^2 - \Lambda \| [c(x)]_+ \| + F_{\rho}^* - F_{\rho}(x) \le 0.$$

It then follows from this and  $\sqrt{a^2 + b^2} \le |a| + |b|$  that

$$\|[c(x)]_+\| \le \frac{\Lambda}{\rho} + \frac{\sqrt{\Lambda^2 + 2\rho(F_\rho(x) - F_\rho^*)}}{\rho} \le \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_\rho(x) - F_\rho^*)}{\rho}}.$$

#### 6.1 Proof for the main results in Section 2

In this subsection, we first establish several technical lemmas and then use them to prove Theorems 1 and 2.

For notational convenience, let

$$f_{\rho}(x) := \mathbb{E}[\tilde{f}(x,\xi)] + \frac{\rho}{2} \| [c(x)]_{+} \|^{2}, \qquad L_{\rho} := L_{\nabla f} + \rho L_{\nabla c^{2}}$$
(48)

for all  $\rho > 0$ . It follows from these, the  $L_{\nabla f}$ -smoothness of  $\mathbb{E}[\tilde{f}(\cdot,\xi)]$ , and the  $L_{\nabla c^2}$ -smoothness of  $||[c(\cdot)]_+||^2/2$  that  $f_{\rho}$  is  $L_{\rho}$ -smooth. One can also observe that  $F_{\rho}(\cdot) = f_{\rho}(\cdot) + \psi(\cdot)$ , where  $F_{\rho}$  is defined in (44). These facts will be used in our subsequent analysis.

The following lemma establishes recursive properties of the quantity  $(\beta_k - 1)\gamma_k(F_{\rho_k}(x_k) - F_{\rho_k}(x)) + ||x - z_k||^2/2$  and its expectation  $\mathbb{E}\left[(\beta_k - 1)\gamma_k(F_{\rho_k}(x_k) - F_{\rho_k}(x)) + ||x - z_k||^2/2\right]$  for all  $x \in \mathcal{X}$  and  $k \ge 1$ .

**Lemma 2.** Suppose that Assumptions 1 and 2 hold. Let  $\{x_k\}$ ,  $\{y_k\}$  and  $\{z_k\}$  be generated in Algorithm 1 with  $\beta_k$ ,  $\gamma_k$ , and  $\rho_k$  satisfying  $\beta_k \ge 1$  and  $L_{\rho_k}\gamma_k - \beta_k < 0$  for all  $k \ge 1$ , where  $L_{\rho_k}$  is defined in (48). Then for all  $x \in \mathcal{X}$  and  $k \ge 1$ , we have

$$\beta_k \gamma_k \big( F_{\rho_k}(x_{k+1}) - F_{\rho_k}(x) \big) + \frac{1}{2} \|x - z_{k+1}\|^2 \le (\beta_k - 1) \gamma_k \big( F_{\rho_k}(x_k) - F_{\rho_k}(x) \big) + \frac{1}{2} \|x - z_k\|^2 + \Delta_k(x),$$
(49)

$$\mathbb{E}\Big[\beta_k \gamma_k \big(F_{\rho_k}(x_{k+1}) - F_{\rho_k}(x)\big) + \frac{1}{2} \|x - z_{k+1}\|^2\Big] \le \mathbb{E}\Big[(\beta_k - 1)\gamma_k \big(F_{\rho_k}(x_k) - F_{\rho_k}(x)\big) + \frac{1}{2} \|x - z_k\|^2\Big] + \frac{\beta_k \gamma_k^2}{2(\beta_k - L_{\rho_k}\gamma_k)} \mathbb{E}[\|\delta_k\|^2],$$
(50)

where  $F_{\rho_k}$  and  $f_{\rho_k}$  are defined in (44) and (48), respectively, and

$$\delta_{k} = g_{k} - \nabla f_{\rho_{k}}(y_{k}), \quad \Delta_{k}(x) = \gamma_{k} \langle \delta_{k}, x - z_{k+1} \rangle - \frac{\beta_{k} - L_{\rho_{k}} \gamma_{k}}{2\beta_{k}} \|z_{k+1} - z_{k}\|^{2}.$$
(51)

*Proof.* Observe from the definitions of  $x_{k+1}$  and  $y_k$  in Algorithm 1 that

$$x_{k+1} - y_k = \beta_k^{-1} (z_{k+1} - z_k).$$
(52)

By this, the definition of  $x_{k+1}$ , and the convexity and  $L_{\rho_k}$ -smoothness of  $f_{\rho_k}$ , one has

$$\begin{aligned} f_{\rho_{k}}(x_{k+1}) &\leq f_{\rho_{k}}(y_{k}) + \langle \nabla f_{\rho_{k}}(y_{k}), x_{k+1} - y_{k} \rangle + \frac{L_{\rho_{k}}}{2} \|x_{k+1} - y_{k}\|^{2} \\ &= f_{\rho_{k}}(y_{k}) + \beta_{k}^{-1} \langle \nabla f_{\rho_{k}}(y_{k}), z_{k+1} - y_{k} \rangle + (1 - \beta_{k}^{-1}) \langle \nabla f_{\rho_{k}}(y_{k}), x_{k} - y_{k} \rangle + \frac{L_{\rho_{k}}}{2} \|x_{k+1} - y_{k}\|^{2} \\ &= \beta_{k}^{-1} \left( f_{\rho_{k}}(y_{k}) + \langle \nabla f_{\rho_{k}}(y_{k}), z_{k+1} - y_{k} \rangle \right) + (1 - \beta_{k}^{-1}) \left( f_{\rho_{k}}(y_{k}) + \langle \nabla f_{\rho_{k}}(y_{k}), x_{k} - y_{k} \rangle \right) + \frac{L_{\rho_{k}}}{2} \|x_{k+1} - y_{k}\|^{2} \\ &\leq \beta_{k}^{-1} \left( f_{\rho_{k}}(y_{k}) + \langle \nabla f_{\rho_{k}}(y_{k}), z_{k+1} - y_{k} \rangle \right) + (1 - \beta_{k}^{-1}) f_{\rho_{k}}(x_{k}) + \frac{L_{\rho_{k}}}{2} \|x_{k+1} - y_{k}\|^{2} \\ &= \beta_{k}^{-1} \left( f_{\rho_{k}}(y_{k}) + \langle g_{k}, z_{k+1} - y_{k} \rangle \right) + \beta_{k}^{-1} \langle \delta_{k}, y_{k} - z_{k+1} \rangle + (1 - \beta_{k}^{-1}) f_{\rho_{k}}(x_{k}) + \frac{L_{\rho_{k}} \beta_{k}^{-2}}{2} \|z_{k+1} - z_{k}\|^{2}, \end{aligned}$$

$$(53)$$

where the first inequality is due to the  $L_{\rho_k}$ -smoothness of  $f_{\rho_k}$ , the first equality follows from the definition of  $x_{k+1}$  in Algorithm 1, and the second inequality is due to  $\beta_k \ge 1$  and the convexity of  $f_{\rho_k}$ , while the last equality follows from (51) and (52).

In addition, by the definition of  $z_{k+1}$  in Algorithm 1, one has

$$z_{k+1} = \arg\min_{x} \left\{ \frac{1}{2\gamma_k} \|x - (z_k - \gamma_k g_k)\|^2 + \psi(x) \right\},\$$

which, together with the strong convexity of its objective function, implies that for all  $x \in \mathcal{X}$ ,

$$\frac{1}{2\gamma_k} \|z_{k+1} - z_k + \gamma_k g_k\|^2 + \psi(z_{k+1}) \le \frac{1}{2\gamma_k} \|x - z_k + \gamma_k g_k\|^2 + \psi(x) - \frac{1}{2\gamma_k} \|x - z_{k+1}\|^2$$

It then follows that

$$\langle g_k, z_{k+1} - y_k \rangle \le \langle g_k, x - y_k \rangle + \psi(x) - \frac{1}{2\gamma_k} \|z_{k+1} - z_k\|^2 + \frac{1}{2\gamma_k} \|x - z_k\|^2 - \frac{1}{2\gamma_k} \|x - z_{k+1}\|^2 - \psi(z_{k+1}).$$

Using this, (51), and the convexity of  $f_{\rho_k}$ , we have

$$\begin{split} f_{\rho_k}(y_k) + \langle g_k, z_{k+1} - y_k \rangle &\leq f_{\rho_k}(y_k) + \langle \nabla f_{\rho_k}(y_k), x - y_k \rangle + \langle \delta_k, x - y_k \rangle + \psi(x) - \frac{1}{2\gamma_k} \|z_{k+1} - z_k\|^2 \\ &+ \frac{1}{2\gamma_k} \|x - z_k\|^2 - \frac{1}{2\gamma_k} \|x - z_{k+1}\|^2 - \psi(z_{k+1}) \\ &\leq F_{\rho_k}(x) + \langle \delta_k, x - y_k \rangle - \frac{1}{2\gamma_k} \|z_{k+1} - z_k\|^2 + \frac{1}{2\gamma_k} \|x - z_k\|^2 - \frac{1}{2\gamma_k} \|x - z_{k+1}\|^2 - \psi(z_{k+1}). \end{split}$$

This together with (53) implies that

$$\begin{split} f_{\rho_k}(x_{k+1}) &\leq \beta_k^{-1} \Big( F_{\rho_k}(x) + \langle \delta_k, x - y_k \rangle - \frac{1}{2\gamma_k} \|z_{k+1} - z_k\|^2 + \frac{1}{2\gamma_k} \|x - z_k\|^2 - \frac{1}{2\gamma_k} \|x - z_{k+1}\|^2 - \psi(z_{k+1}) \Big) \\ &+ (1 - \beta_k^{-1}) f_{\rho_k}(x_k) + \beta_k^{-1} \langle \delta_k, y_k - z_{k+1} \rangle + \frac{L_{\rho_k} \beta_k^{-2}}{2} \|z_{k+1} - z_k\|^2 \\ &= \beta_k^{-1} F_{\rho_k}(x) + (1 - \beta_k^{-1}) f_{\rho_k}(x_k) + \beta_k^{-1} \langle \delta_k, x - z_{k+1} \rangle + \frac{1}{2} \Big( L_{\rho_k} \beta_k^{-2} - \frac{\beta_k^{-1}}{\gamma_k} \Big) \|z_{k+1} - z_k\|^2 \\ &+ \frac{\beta_k^{-1}}{2\gamma_k} \|x - z_k\|^2 - \frac{\beta_k^{-1}}{2\gamma_k} \|x - z_{k+1}\|^2 - \beta_k^{-1} \psi(z_{k+1}). \end{split}$$

Also, by  $\beta_k \geq 1$ , the definition of  $x_{k+1}$ , and the convexity of  $\psi$ , one has

$$\psi(x_{k+1}) = \psi\left(\beta_k^{-1} z_{k+1} + (1 - \beta_k^{-1}) x_k\right) \le \beta_k^{-1} \psi(z_{k+1}) + (1 - \beta_k^{-1}) \psi(x_k).$$

Summing up the above two inequalities gives

$$F_{\rho_k}(x_{k+1}) \leq \beta_k^{-1} F_{\rho_k}(x) + (1 - \beta_k^{-1}) F_{\rho_k}(x_k) + \beta_k^{-1} \langle \delta_k, x - z_{k+1} \rangle + \frac{1}{2} \Big( L_{\rho_k} \beta_k^{-2} - \frac{\beta_k^{-1}}{\gamma_k} \Big) \|z_{k+1} - z_k\|^2 \\ + \frac{\beta_k^{-1}}{2\gamma_k} \|x - z_k\|^2 - \frac{\beta_k^{-1}}{2\gamma_k} \|x - z_{k+1}\|^2.$$

Multiplying this by  $\beta_k \gamma_k$  and rearranging the terms lead to the conclusion (49).

We next show that (50) holds. To this end, one can first observe that

$$\Delta_{k}(x) \stackrel{(51)}{=} \gamma_{k} \langle \delta_{k}, x - z_{k} \rangle + \gamma_{k} \langle \delta_{k}, z_{k+1} - z_{k} \rangle - \frac{\beta_{k} - L_{\rho_{k}} \gamma_{k}}{2\beta_{k}} \| z_{k+1} - z_{k} \|^{2}$$

$$\leq \gamma_{k} \langle \delta_{k}, x - z_{k} \rangle + \frac{\beta_{k} \gamma_{k}^{2} \| \delta_{k} \|^{2}}{2(\beta_{k} - L_{\rho_{k}} \gamma_{k})}, \qquad (54)$$

where the inequality follows from  $\beta_k - L_{\rho_k} \gamma_k > 0$  and Young's inequality. Let  $\Xi_k = \{\xi_1, \xi_2, \ldots, \xi_{k-1}\}$ denote the collection of samples drawn up to iteration k-1 in Algorithm 1. It then follows from (51),  $g_k = \nabla \tilde{f}(y_k, \xi_k) + \rho_k \nabla c(y_k) [c(y_k)]_+, y_k \in \mathcal{X}$ , and Assumption 2 that for all  $x \in \mathcal{X}$ ,

$$\mathbb{E}[\langle \delta_k, x - z_k \rangle | \Xi_k] = \mathbb{E}[\langle \nabla \tilde{f}(y_k, \xi_k) - \nabla f(y_k), x - z_k \rangle | \Xi_k] = \langle \mathbb{E}[\nabla \tilde{f}(y_k, \xi_k) | \Xi_k] - \nabla f(y_k), x - z_k \rangle = 0.$$

This and (54) immediately imply that

$$\mathbb{E}[\Delta_k(x)|\Xi_k] \le \frac{\beta_k \gamma_k^2 \mathbb{E}[\|\delta_k\|^2 |\Xi_k]}{2(\beta_k - L_{\rho_k} \gamma_k)}.$$

Using this and taking expectation on both sides of (49) conditional on  $\Xi_k$ , we obtain

$$\mathbb{E}\Big[\beta_k \gamma_k(F_{\rho_k}(x_{k+1}) - F_{\rho_k}(x)) + \frac{1}{2} \|x - z_{k+1}\|^2 \Big| \Xi_k \Big] \le \mathbb{E}\Big[(\beta_k - 1)\gamma_k(F_{\rho_k}(x_k) - F_{\rho_k}(x)) + \frac{1}{2} \|x - z_k\|^2 \Big| \Xi_k \Big] \\
+ \frac{\beta_k \gamma_k^2 \mathbb{E}[\|\delta_k\|^2 |\Xi_k]}{2(\beta_k - L_{\rho_k} \gamma_k)}.$$

The conclusion (50) then follows from taking expectation on both sides of this inequality.

We next derive upper bounds for  $F_{\rho}(x_k) - F_{\rho}^*$  and  $\mathbb{E}[F_{\rho}(x_k) - F_{\rho}^*]$  under the setting where constant penalty parameter  $\rho_k \equiv \rho$  is used in Algorithm 1 for some  $\rho > 0$ .

**Lemma 3.** Suppose that Assumptions 1 and 2 hold. Let  $\{x_k\}$  be generated in Algorithm 1 with  $\rho_k \equiv \rho$ , and  $\beta_k \in [1, \infty)$  and  $\gamma_k > 0$  satisfying  $\beta_1 = 1$ ,  $0 < (\beta_{k+1} - 1)\gamma_{k+1} \leq \beta_k \gamma_k$ , and  $2L_{\rho}\gamma_k \leq \beta_k$  for some  $\rho > 0$  and all  $k \geq 1$ , where  $L_{\rho}$  is defined in (48). Then for all  $k \geq 2$ , we have

$$F_{\rho}(x_{k}) - F_{\rho}^{*} \leq \frac{1}{(\beta_{k} - 1)\gamma_{k}} \Big( \frac{1}{2} D_{\mathcal{X}}^{2} + 2GD_{\mathcal{X}} \sum_{i=1}^{k-1} \gamma_{i} \Big),$$
$$\mathbb{E}[F_{\rho}(x_{k}) - F_{\rho}^{*}] \leq \frac{1}{(\beta_{k} - 1)\gamma_{k}} \Big( \frac{1}{2} D_{\mathcal{X}}^{2} + \sigma^{2} \sum_{i=1}^{k-1} \gamma_{i}^{2} \Big),$$

where  $F_{\rho}$  and  $F_{\rho}^*$  are defined in (44), and  $D_{\mathcal{X}}$ ,  $\sigma$  and G are given in Assumptions 1 and 2, respectively.

Proof. Since  $2L_{\rho}\gamma_k \leq \beta_k$  and  $\beta_k \geq 1$  for all k, the assumption of Lemma 2 holds. Let  $x_{\rho}^* \in \arg\min_x F_{\rho}(x)$ . It follows from this and (44) that  $F_{\rho}^* = F_{\rho}(x_{\rho}^*)$  and  $F_{\rho}(x_k) \geq F_{\rho}^*$  for all k. By these, (49) with  $x = x_{\rho}^*$ , (51), and the assumption that  $\rho_k = \rho$  and  $0 < (\beta_{k+1} - 1)\gamma_{k+1} \leq \beta_k \gamma_k$  for all k, one has that

$$(\beta_{i+1}-1)\gamma_{i+1}(F_{\rho}(x_{i+1})-F_{\rho}^{*})+\frac{1}{2}\|x_{\rho}^{*}-z_{i+1}\|^{2} \leq \beta_{i}\gamma_{i}(F_{\rho}(x_{i+1})-F_{\rho}^{*})+\frac{1}{2}\|x_{\rho}^{*}-z_{i+1}\|^{2}$$

$$\leq (\beta_{i}-1)\gamma_{i}(F_{\rho}(x_{i})-F_{\rho}^{*})+\frac{1}{2}\|x_{\rho}^{*}-z_{i}\|^{2}+\gamma_{i}\langle\delta_{i},x_{\rho}^{*}-z_{i+1}\rangle-\frac{\beta_{i}-L_{\rho}\gamma_{i}}{2\beta_{i}}\|z_{i+1}-z_{i}\|^{2}$$

$$\leq (\beta_{i}-1)\gamma_{i}(F_{\rho}(x_{i})-F_{\rho}^{*})+\frac{1}{2}\|x_{\rho}^{*}-z_{i}\|^{2}+\gamma_{i}\|\delta_{i}\|\|x_{\rho}^{*}-z_{i+1}\| \quad \forall i \geq 1.$$
(55)

In addition, by (51) and Assumption 2, one can observe that

$$\|\delta_i\| \stackrel{(51)}{=} \|\nabla \tilde{f}(y_i,\xi_i) - \mathbb{E}[\nabla \tilde{f}(y_i,\xi)]\| \le \|\nabla \tilde{f}(y_i,\xi_i)\| + \mathbb{E}[\|\nabla \tilde{f}(y_i,\xi)\|] \le 2G.$$
(56)

Using this,  $\beta_1 = 1$ ,  $||x_{\rho}^* - z_1|| \leq D_{\mathcal{X}}$ , and summing up (55) from i = 1 to k - 1, we obtain that

$$\begin{aligned} (\beta_k - 1)\gamma_k(F_{\rho}(x_k) - F_{\rho}^*) &\leq (\beta_1 - 1)\gamma_1(F_{\rho}(x_1) - F_{\rho}^*) + \frac{1}{2} \|x_{\rho}^* - z_1\|^2 + \sum_{i=1}^{k-1} \gamma_i \|\delta_i\| \|x_{\rho}^* - z_i\| \\ &\leq \frac{D_{\mathcal{X}}^2}{2} + 2GD_{\mathcal{X}} \sum_{i=1}^{k-1} \gamma_i, \end{aligned}$$

which implies that the first conclusion of this lemma holds.

We next show that the second conclusion of this lemma holds. Indeed, by  $2L_{\rho}\gamma_k \leq \beta_k$  and  $\beta_k \geq 1$ , one has  $\beta_k/(\beta_k - L_{\rho}\gamma_k) \leq 2$ . Using this, (50),  $0 < (\beta_{k+1} - 1)\gamma_{k+1} \leq \beta_k\gamma_k$ , and  $F_{\rho}(x) \geq F_{\rho}^*$ , we obtain that

$$\mathbb{E}\Big[(\beta_{i+1}-1)\gamma_{i+1}(F_{\rho}(x_{i+1})-F_{\rho}^{*})+\frac{1}{2}\|x_{\rho}^{*}-z_{i+1}\|^{2}\Big] \leq \mathbb{E}\Big[\beta_{i}\gamma_{i}(F_{\rho}(x_{i+1})-F_{\rho}^{*})+\frac{1}{2}\|x_{\rho}^{*}-z_{i+1}\|^{2}\Big] \\
\leq \mathbb{E}\Big[(\beta_{i}-1)\gamma_{i}(F_{\rho}(x_{i})-F_{\rho}^{*})+\frac{1}{2}\|x_{\rho}^{*}-z_{i}\|^{2}\Big]+\gamma_{i}^{2}\mathbb{E}[\|\delta_{i}\|^{2}],$$

where the second inequality follows from (50) and  $\beta_i/(\beta_i - L_\rho \gamma_i) \leq 2$ . Using  $\beta_1 = 1$ ,  $\mathbb{E}[\|\delta_i\|^2] \leq \sigma^2$ ,  $\|x_\rho^* - z_1\| \leq D_X$ , and summing up the above inequalities from i = 1 to k - 1, we have

$$(\beta_k - 1)\gamma_k \mathbb{E}[F_{\rho}(x_k) - F_{\rho}^*] \le \frac{1}{2} \|x_{\rho}^* - z_1\|^2 + \sum_{i=1}^{k-1} \gamma_i^2 \mathbb{E}[\|\delta_i\|^2] \le \frac{1}{2} D_{\mathcal{X}}^2 + \sigma^2 \sum_{i=1}^{k-1} \gamma_i^2,$$

which implies that the second conclusion of this lemma holds.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Notice from (9) that  $\rho_k = \rho$ ,  $\beta_k = (k+1)/2$ , and  $\gamma_k = (k+1)/(4L_{\rho})$ , where  $L_{\rho} = L_{\nabla f} + \rho L_{\nabla c^2}$  and  $\rho = K^{3/2}$ . Observe that  $0 < ((k+2)/2 - 1)(k+2) \le (k+1)^2/2$  for all  $k \ge 1$ , which implies  $0 < (\beta_{k+1} - 1)\gamma_{k+1} \le \beta_k \gamma_k$ . In addition,  $2L_{\rho}\gamma_k \le \beta_k$  clearly holds. Hence, the assumption of Lemma 3 holds for such  $\rho_k$ ,  $\beta_k$ , and  $\gamma_k$ . It then follows from  $K \ge 2$  and Lemma 3 with such  $\rho_k$ ,  $\beta_k$ , and  $\gamma_k$  that

$$F_{\rho}(x_{K}) - F_{\rho}^{*} \leq \frac{1}{(\beta_{K} - 1)\gamma_{K}} \left(\frac{1}{2}D_{\mathcal{X}}^{2} + 2GD_{\mathcal{X}}\sum_{i=1}^{K-1}\gamma_{i}\right)$$

$$= \frac{4L_{\rho}}{((K+1)/2 - 1)(K+1)} \left(\frac{1}{2}D_{\mathcal{X}}^{2} + 2GD_{\mathcal{X}}\sum_{i=1}^{K-1}\frac{i+1}{4L_{\rho}}\right)$$

$$= \frac{4L_{\rho}D_{\mathcal{X}}^{2}}{K^{2} - 1} + \frac{2GD_{\mathcal{X}}(K+2)}{K+1} \leq \frac{4L_{\rho}D_{\mathcal{X}}^{2}}{K^{2} - 1} + \frac{8GD_{\mathcal{X}}}{3},$$
(57)

where the last inequality is due to  $K \ge 2$ . By (46) and (57), one has

$$\begin{split} \|[c(x_K)]_+\| & \stackrel{(46)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_\rho(x_K) - F_\rho^*)}{\rho}} \stackrel{(57)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{2}{\rho} \left(\frac{4L_\rho D_{\mathcal{X}}^2}{K^2 - 1} + \frac{8GD_{\mathcal{X}}}{3}\right)} \\ & \leq \frac{2\Lambda}{\rho} + \sqrt{\frac{8L_\rho D_{\mathcal{X}}^2}{\rho(K^2 - 1)}} + \sqrt{\frac{16GD_{\mathcal{X}}}{3\rho}} = \frac{2\Lambda}{K^{3/2}} + \sqrt{\frac{8(L_{\nabla f} + L_{\nabla c^2}K^{3/2})D_{\mathcal{X}}^2}{K^{3/2}(K^2 - 1)}} + \sqrt{\frac{16GD_{\mathcal{X}}}{3K^{3/2}}} \\ & \leq \frac{2\Lambda}{K^{3/2}} + \sqrt{\frac{8L_{\nabla f} D_{\mathcal{X}}^2}{K^{3/2}(K^2 - 1)}} + \sqrt{\frac{8L_{\nabla c^2} D_{\mathcal{X}}^2}{K^2 - 1}} + \sqrt{\frac{16GD_{\mathcal{X}}}{3K^{3/2}}}, \end{split}$$

which, along with  $K^2 - 1 \ge 3K^2/4$  due to  $K \ge 2$ , implies that (10) holds. In addition, using  $\rho = K^{3/2}$ ,  $L_{\rho} = L_{\nabla f} + \rho L_{\nabla c^2}$ ,  $K \ge 2$ , and Lemma 3, we have that

$$\mathbb{E}[F_{\rho}(x_{K}) - F_{\rho}^{*}] \stackrel{\text{Lemma 3}}{\leq} \frac{1}{(\beta_{K} - 1)\gamma_{K}} \left(\frac{1}{2}D_{\mathcal{X}}^{2} + \sigma^{2}\sum_{i=1}^{K-1}\gamma_{i}^{2}\right) \\
= \frac{4L_{\rho}}{((K+1)/2 - 1)(K+1)} \left(\frac{1}{2}D_{\mathcal{X}}^{2} + \sigma^{2}\sum_{i=1}^{K-1} \left(\frac{i+1}{4L_{\rho}}\right)^{2}\right) \leq \frac{4L_{\rho}D_{\mathcal{X}}^{2}}{K^{2} - 1} + \frac{\sigma^{2}(K+1)^{3}}{6L_{\rho}(K^{2} - 1)} \\
= \frac{4L_{\nabla f}D_{\mathcal{X}}^{2}}{K^{2} - 1} + \frac{4L_{\nabla c^{2}}D_{\mathcal{X}}^{2}K^{3/2}}{K^{2} - 1} + \frac{\sigma^{2}(K+1)^{2}}{2(L_{\nabla f} + L_{\nabla c^{2}}K^{3/2})(K-1)} \\
\leq \frac{16L_{\nabla f}D_{\mathcal{X}}^{2}}{3K^{2}} + \frac{16L_{\nabla c^{2}}D_{\mathcal{X}}^{2}}{3K^{1/2}} + \frac{9\sigma^{2}}{4L_{\nabla c^{2}}K^{1/2}} \leq \frac{6L_{\nabla f}D_{\mathcal{X}}^{2}}{K^{2}} + \frac{6L_{\nabla c^{2}}D_{\mathcal{X}}^{2} + 3\sigma^{2}L_{\nabla c^{2}}^{-1}}{K^{1/2}},$$
(58)

where the second inequality follows from the fact that  $\sum_{i=1}^{K-1} (i+1)^2 \leq \int_1^K (\tau+1)^2 d\tau \leq (K+1)^3/3$ , and the third inequality follows from the fact that  $K^2 - 1 \geq 3K^2/4$ ,  $K + 1 \leq 3K/2$ , and  $K - 1 \geq K/2$ due to  $K \geq 2$ . The relation (11) then follows from (58) and  $\mathbb{E}[F(x_K) - F^*] \leq \mathbb{E}[F_{\rho}(x_K) - F_{\rho}^*]$  due to (45).

Lastly, it follows from  $\rho = K^{3/2}$ , (46), and (58) that

$$\mathbb{E}[\|[c(x_K)]_+\|] \stackrel{(46)}{\leq} \frac{2\Lambda}{\rho} + \mathbb{E}\left[\sqrt{\frac{2F_{\rho}(x_K) - F_{\rho}^*}{\rho}}\right] \leq \frac{2\Lambda}{\rho} + \sqrt{\frac{2\mathbb{E}\left[F_{\rho}(x_K) - F_{\rho}^*\right]}{\rho}} \\ \stackrel{(58)}{\leq} \frac{2\Lambda}{K^{3/2}} + \sqrt{\frac{12L_{\nabla f}D_{\mathcal{X}}^2}{K^{7/2}}} + \sqrt{12L_{\nabla c^2}D_{\mathcal{X}}^2 + 6\sigma^2 L_{\nabla c^2}^{-1}} \frac{1}{K},$$

which together with (45) implies that (12) holds.

In the remainder of this subsection, we provide a proof of Theorem 2. Before proceeding, we first derive an upper bound on  $F_{\rho_k}(x_k) - F^*_{\rho_k}$  and  $\mathbb{E}[F_{\rho_k}(x_k) - F^*_{\rho_k}]$  for all  $k \ge 2$ .

**Lemma 4.** Suppose that Assumptions 1 and 2 hold. Let  $\{x_k\}$  be generated in Algorithm 1 with a non-decreasing positive  $\{\rho_k\}$ ,  $\beta_k \in [1, \infty)$  and  $\gamma_k > 0$  satisfying  $\beta_1 = 1$ ,  $0 < (\beta_{k+1} - 1)\gamma_{k+1} \le \beta_k \gamma_k (1 - 2(\rho_{k+1} - \rho_k)/\rho_{k+1})$ , and  $2L_{\rho_k}\gamma_k \le \beta_k$  for all  $k \ge 1$ , where  $L_{\rho_k}$  is defined in (48). Then for all  $k \ge 2$ , we have

$$F_{\rho_k}(x_k) - F_{\rho_k}^* \le \frac{1}{(\beta_k - 1)\gamma_k} \Big( \frac{1}{2} D_{\mathcal{X}}^2 + 2GD_{\mathcal{X}} \sum_{i=1}^{k-1} \gamma_i + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} \Big),$$
$$\mathbb{E} \Big[ F_{\rho_k}(x_k) - F_{\rho_k}^* \Big] \le \frac{1}{(\beta_k - 1)\gamma_k} \Big( \frac{1}{2} D_{\mathcal{X}}^2 + \sigma^2 \sum_{i=1}^{k-1} \gamma_i^2 + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} \Big),$$

where  $F_{\rho_k}$  and  $F^*_{\rho_k}$  are defined in (44), and  $D_{\mathcal{X}}$ ,  $\Lambda$ ,  $\sigma$  and G are given in Assumptions 1 and 2, respectively.

Proof. Since  $2L_{\rho_k}\gamma_k \leq \beta_k$  and  $\beta_k \geq 1$  for all  $k \geq 1$ , the assumption of Lemma 2 holds. Let  $x^*$  and  $x_k^*$  be an arbitrary optimal solution of problems (1) and (8), respectively. It then follows that  $F_{\rho_k}^* = F_{\rho_k}(x_k^*)$ ,  $F_{\rho_k}(x_k) \geq F_{\rho_k}^*$ ,  $F_{\rho_{k+1}}^* \geq F_{\rho_k}^*$ , and  $F_{\rho_k}(x^*) = F^*$  for all  $k \geq 1$ , where  $F^*$  is defined in (1). By these, (45), (46), and (49) with  $x = x^*$ , one has that for all  $k \geq 1$ ,

$$\frac{1}{2} \|x^{*} - z_{k}\|^{2} - \frac{1}{2} \|x^{*} - z_{k+1}\|^{2} + \Delta_{k}(x) \stackrel{(49)}{\geq} \beta_{k}\gamma_{k}(F_{\rho_{k}}(x_{k+1}) - F^{*}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}) \\
= \beta_{k}\gamma_{k}(F_{\rho_{k}}(x_{k+1}) - F^{*}_{\rho_{k+1}}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}_{\rho_{k}}) + \beta_{k}\gamma_{k}(F^{*}_{\rho_{k+1}} - F^{*}) - (\beta_{k} - 1)\gamma_{k}(F^{*}_{\rho_{k}} - F^{*}) \\
\geq \beta_{k}\gamma_{k}(F_{\rho_{k}}(x_{k+1}) - F^{*}_{\rho_{k+1}}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}_{\rho_{k}}) + \gamma_{k}(F^{*}_{k} - F^{*}) \\
= \beta_{k}\gamma_{k}(F_{\rho_{k}}(x_{k+1}) - F^{*}_{\rho_{k+1}}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}_{\rho_{k}}) + \gamma_{k}(F(x^{*}_{k}) - F^{*}) + \frac{\gamma_{k}\rho_{k}}{2} ||[c(x^{*}_{k})]_{+}||^{2} \\
\stackrel{(45)}{\leq} \beta_{k}\gamma_{k}(F_{\rho_{k}}(x_{k+1}) - F^{*}_{\rho_{k+1}}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}_{\rho_{k}}) - \gamma_{k}\Lambda ||[c(x^{*}_{k})]_{+}|| + \frac{\gamma_{k}\rho_{k}}{2} ||[c(x^{*}_{k})]_{+}||^{2} \\
\geq \beta_{k}\gamma_{k}(F_{\rho_{k}}(x_{k+1}) - F^{*}_{\rho_{k+1}}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}_{\rho_{k}}) - \frac{\gamma_{k}\Lambda^{2}}{2\rho_{k}} \\
= \beta_{k}\gamma_{k}(F_{\rho_{k+1}}(x_{k+1}) - F^{*}_{\rho_{k+1}}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}_{\rho_{k}}) - \frac{\gamma_{k}\Lambda^{2}}{2\rho_{k}} \\
- \frac{1}{2}\beta_{k}\gamma_{k}(\rho_{k+1} - \rho_{k})\left(\frac{2\Lambda}{\rho_{k+1}} + \sqrt{\frac{2(F_{\rho_{k+1}}(x_{k+1}) - F^{*}_{\rho_{k+1}})}{\rho_{k+1}}}) - (\beta_{k} - 1)\gamma_{k}(F_{\rho_{k}}(x_{k}) - F^{*}_{\rho_{k}}) - \frac{\gamma_{k}\Lambda^{2}}{2\rho_{k}} \\
- \frac{4\beta_{k}\gamma_{k}(\rho_{k+1} - \rho_{k})\Lambda^{2}}{\rho_{k+1}^{2}},$$
(59)

where the second inequality is due to  $F_{\rho_{k+1}}^* \ge F_{\rho_k}^*$ , the second equality follows from (8) and the definition of F in (1), and the fourth inequality is due to Young's inequality, while the last inequality follows from the fact that  $(a + b)^2 \le 2(a^2 + b^2)$  for all  $a, b \in \mathbb{R}$ .

Recall that  $0 < (\beta_{k+1} - 1)\gamma_{k+1} \le \beta_k \gamma_k (1 - 2(\rho_{k+1} - \rho_k)/\rho_{k+1})$  and  $L_{\rho_k} \gamma_k - \beta_k < 0$  for all  $k \ge 1$ . These, together with (51), (59), and  $F_{\rho_i}(x_i) \ge F_{\rho_i}^*$  for all i, imply that for all  $i \ge 1$ ,

$$\begin{aligned} (\beta_{i+1} - 1)\gamma_{i+1}(F_{\rho_{i+1}}(x_{i+1}) - F_{\rho_{i+1}}^{*}) &+ \frac{1}{2} \|x^{*} - z_{i+1}\|^{2} \\ &\leq \beta_{i}\gamma_{i} \Big(1 - \frac{2(\rho_{i+1} - \rho_{i})}{\rho_{i+1}}\Big) (F_{\rho_{i+1}}(x_{i+1}) - F_{\rho_{i+1}}^{*}) + \frac{1}{2} \|x^{*} - z_{i+1}\|^{2} \\ &\stackrel{(51)(59)}{\leq} (\beta_{i} - 1)\gamma_{i}(F_{\rho_{i}}(x_{i}) - F_{\rho_{i}}^{*}) + \frac{1}{2} \|x^{*} - z_{i}\|^{2} + \gamma_{i}\langle\delta_{i}, x^{*} - z_{i+1}\rangle - \frac{\beta_{i} - L_{\rho_{i}}\gamma_{i}}{2\beta_{i}} \|z_{i+1} - z_{i}\|^{2} \\ &+ \frac{\gamma_{i}\Lambda^{2}}{2\rho_{i}} + \frac{4\beta_{i}\gamma_{i}(\rho_{i+1} - \rho_{i})\Lambda^{2}}{\rho_{i+1}^{2}} \\ &\leq (\beta_{i} - 1)\gamma_{i}(F_{\rho_{i}}(x_{i}) - F_{\rho_{i}}^{*}) + \frac{1}{2} \|x^{*} - z_{i}\|^{2} + \gamma_{i}\|\delta_{i}\|\|x^{*} - z_{i+1}\| + \frac{\gamma_{i}\Lambda^{2}}{2\rho_{i}} + \frac{4\beta_{i}\gamma_{i}(\rho_{i+1} - \rho_{i})\Lambda^{2}}{\rho_{i+1}^{2}}. \end{aligned}$$
(60)

Using (56),  $\beta_1 = 1$ ,  $||x^* - z_1|| \le D_{\mathcal{X}}$ , and summing up (60) from i = 1 to k - 1, we obtain that

$$\begin{aligned} &(\beta_{k}-1)\gamma_{k}(F_{\rho_{k}}(x_{k})-F_{\rho_{k}}^{*})\\ &\leq (\beta_{1}-1)\gamma_{1}(F_{\rho_{k}}(x_{1})-F_{\rho_{k}}^{*})+\frac{1}{2}\|x^{*}-z_{1}\|^{2}+\sum_{i=1}^{k-1}\gamma_{i}\|\delta_{i}\|\|x^{*}-z_{i}\|+\frac{\Lambda^{2}}{2}\sum_{i=1}^{k-1}\frac{\gamma_{i}}{\rho_{i}}+4\Lambda^{2}\sum_{i=1}^{k-1}\frac{\beta_{i}\gamma_{i}(\rho_{i+1}-\rho_{i})}{\rho_{i+1}^{2}}\\ &\leq \frac{D_{\mathcal{X}}^{2}}{2}+2GD_{\mathcal{X}}\sum_{i=1}^{k-1}\gamma_{i}+\frac{\Lambda^{2}}{2}\sum_{i=1}^{k-1}\frac{\gamma_{i}}{\rho_{i}}+4\Lambda^{2}\sum_{i=1}^{k-1}\frac{\beta_{i}\gamma_{i}(\rho_{i+1}-\rho_{i})}{\rho_{i+1}^{2}},\end{aligned}$$

which, along with the assumption that  $(\beta_k - 1)\gamma_k > 0$  for all k > 1, implies that the first conclusion of this lemma holds.

We next prove that the second conclusion of this lemma holds. Using (50) and applying similar arguments as those used to derive (59), we can show that for all  $k \ge 1$ ,

$$\frac{1}{2}\mathbb{E}[\|x^{*} - z_{k}\|^{2}] - \frac{1}{2}\mathbb{E}[\|x^{*} - z_{k+1}\|^{2}] + \frac{\beta_{k}\gamma_{k}^{2}}{2(\beta_{k} - L_{\rho_{k}}\gamma_{k})}\mathbb{E}[\|\delta_{k}\|^{2}]$$

$$\geq \beta_{k}\gamma_{k}\left(1 - \frac{2(\rho_{k+1} - \rho_{k})}{\rho_{k+1}}\right)\mathbb{E}[F_{\rho_{k+1}}(x_{k+1}) - F_{\rho_{k+1}}^{*}] - (\beta_{k} - 1)\gamma_{k}\mathbb{E}[(F_{\rho_{k}}(x_{k}) - F_{\rho_{k}}^{*}] - \frac{\gamma_{k}\Lambda^{2}}{2\rho_{k}} - \frac{4\beta_{k}\gamma_{k}(\rho_{k+1} - \rho_{k})\Lambda^{2}}{\rho_{k+1}^{2}}.$$
(61)

Also, by the assumption  $2L_{\rho_k}\gamma_k \leq \beta_k$ , one has  $\beta_k/(\beta_k - L_{\rho_k}\gamma_k) \leq 2$ . Using this, (61), and the assumption that  $0 < (\beta_{k+1} - 1)\gamma_{k+1} \leq \beta_k\gamma_k(1 - 2(\rho_{k+1} - \rho_k)/\rho_{k+1})$  for all k, we obtain that for all  $i \geq 1$ ,

$$\begin{aligned} &(\beta_{i+1}-1)\gamma_{i+1}\mathbb{E}[F_{\rho_{i+1}}(x_{i+1})-F_{\rho_{i+1}}^*]+\frac{1}{2}\mathbb{E}[\|x^*-z_{i+1}\|^2]\\ &\leq \beta_i\gamma_i\Big(1-\frac{2(\rho_{i+1}-\rho_i)}{\rho_{i+1}}\Big)\mathbb{E}[F_{\rho_{i+1}}(x_{i+1})-F_{\rho_{i+1}}^*]+\frac{1}{2}\mathbb{E}[\|x^*-z_{i+1}\|^2]\\ &\leq (\beta_i-1)\gamma_i\mathbb{E}[F_{\rho_i}(x_i)-F_{\rho_i}^*]+\frac{1}{2}\mathbb{E}[\|x^*-z_i\|^2]+\gamma_i^2\mathbb{E}[\|\delta_i\|^2]+\frac{\gamma_i\Lambda^2}{2\rho_i}+\frac{4\beta_i\gamma_i(\rho_{i+1}-\rho_i)\Lambda^2}{\rho_{i+1}^2}.\end{aligned}$$

Using  $\beta_1 = 1$ ,  $\mathbb{E}[\|\delta_i\|^2] \leq \sigma^2$ ,  $\|x^* - z_1\| \leq D_{\mathcal{X}}$ , and summing up the above inequalities from i = 1 to k - 1, we have

$$(\beta_k - 1)\gamma_k \mathbb{E}[F_{\rho_k}(x_k) - F_{\rho_k}^*] \le \frac{1}{2}D_{\mathcal{X}}^2 + \sigma^2 \sum_{i=1}^{k-1} \gamma_i^2 + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_i^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_i^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_i - \rho_i)}{\rho_i^$$

which, along with the assumption that  $(\beta_k - 1)\gamma_k > 0$  for all k > 1, implies that the second conclusion of this lemma holds.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** Notice that  $\rho_k = (k+4)^{3/2}$ ,  $\beta_k = (k+4)/5$ ,  $\gamma_k = (k+4)/(10L_{\rho_k})$ , and  $L_{\rho_k} = L_{\nabla f} + (k+4)^{3/2}L_{\nabla c^2}$ . Also, one can verify that  $3(k+4)^2 \leq (k+5)((k+4)^2 - k(k+5))$ , and

$$(k+5)^{3/2} - (k+4)^{3/2} \le \frac{3}{2}(k+5)^{1/2}$$
 (due to the convexity of  $\tau^{3/2}$ ). (62)

Using these and  $L_{\rho_{k+1}} \ge L_{\rho_k}$ , we have

$$\frac{2\beta_k\gamma_k(\rho_{k+1}-\rho_k)}{\rho_{k+1}} = \frac{2(k+4)^2((k+5)^{3/2}-(k+4)^{3/2})}{50L_{\rho_k}(k+5)^{3/2}} \le \frac{3(k+4)^2}{50L_{\rho_k}(k+5)} \le \frac{(k+4)^2}{50L_{\rho_k}} - \frac{k(k+5)}{50L_{\rho_k}} \le \frac{(k+4)^2}{50L_{\rho_k}} - \frac{k(k+5)}{50L_{\rho_{k+1}}} = \beta_k\gamma_k - (\beta_{k+1}-1)\gamma_{k+1}.$$

Hence, the assumption of Lemma 4 holds for such  $\rho_k$ ,  $\beta_k$ , and  $\gamma_k$ . In addition, one can verify that

$$\sum_{i=1}^{k-1} (i+4)^{-1/2} \le 2(k+4)^{1/2}, \quad \sum_{i=1}^{k-1} (i+4)^{-2} \le 1, \quad \sum_{i=1}^{k-1} (i+4)^{-1} \le \log(k+4).$$
(63)

By (62), (63), Lemma 4, and the definitions of  $C_1$ ,  $C_2$  in (17), one has that for all  $k \ge 2$ ,

$$\begin{split} F_{\rho_k}(x_k) - F_{\rho_k}^* &\leq \frac{1}{(\beta_k - 1)\gamma_k} \Big( \frac{1}{2} D_{\mathcal{X}}^2 + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + 2GD_{\mathcal{X}} \sum_{i=1}^{k-1} \gamma_i \Big) \\ &= \frac{50L_{\rho_k}}{(k-1)(k+4)} \Big( \frac{1}{2} D_{\mathcal{X}}^2 + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{i+4}{10(L_{\nabla f} + (i+4)^{3/2}L_{\nabla c^2})(i+4)^{3/2}} \\ &\quad + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{(i+4)^2((i+5)^{3/2} - (i+4)^{3/2})}{50(L_{\nabla f} + (i+4)^{3/2}L_{\nabla c^2})(i+5)^3} + 2GD_{\mathcal{X}} \sum_{i=1}^{k-1} \frac{i+4}{10(L_{\nabla f} + (i+4)^{3/2}L_{\nabla c^2})} \Big) \\ &\stackrel{(62)}{\leq} \frac{50L_{\rho_k}}{(k-1)(k+4)} \Big( \frac{1}{2} D_{\mathcal{X}}^2 + \frac{\Lambda^2}{20L_{\nabla c^2}} \sum_{i=1}^{k-1} \frac{1}{(i+4)^2} + \frac{3\Lambda^2}{25L_{\nabla c^2}} \sum_{i=1}^{k-1} \frac{1}{(i+4)^2} + \frac{GD_{\mathcal{X}}}{5L_{\nabla c^2}} \sum_{i=1}^{k-1} \frac{1}{(i+4)^{1/2}} \Big) \\ &\stackrel{(63)}{\leq} \frac{50L_{\rho_k}}{(k-1)(k+4)} \Big( \frac{1}{2} D_{\mathcal{X}}^2 + \frac{\Lambda^2}{20L_{\nabla c^2}} + \frac{3\Lambda^2}{25L_{\nabla c^2}} + \frac{2GD_{\mathcal{X}}}{5L_{\nabla c^2}} (k+4)^{\frac{1}{2}} \Big) \\ &\stackrel{(17)}{\leq} \frac{C_1L_{\rho_k}}{(k-1)(k+4)} + \frac{20GD_{\mathcal{X}} (L_{\nabla f} + (k+4)^{3/2}L_{\nabla c^2})(k+4)^{1/2}}{L_{\nabla c^2}(k-1)(k+4)} \leq \frac{C_1L_{\rho_k}}{(k-1)(k+4)} + C_2, \end{split}$$

where the last inequality follows from  $k \ge 2$  and (17). Applying similar arguments as those used to derive (64), we can show that for all  $k \ge 2$ ,

$$\mathbb{E}[F_{\rho_k}(x_k) - F_{\rho_k}^*] \leq \frac{1}{(\beta_k - 1)\gamma_k} \Big( \frac{1}{2} D_{\mathcal{X}}^2 + \sigma^2 \sum_{i=1}^{k-1} \gamma_i^2 + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{\beta_i \gamma_i (\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i}{\rho_i} \Big)$$

$$\leq \frac{C_1 L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\rho_k}}{2(k-1)(k+4)} \sum_{i=1}^{k-1} \frac{(i+4)^2}{(L_{\nabla f} + (i+4)^{3/2} L_{\nabla c^2})^2}$$

$$\leq \frac{C_1 L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\rho_k}}{2L_{\nabla c^2}^2(k-1)(k+4)} \sum_{i=1}^{k-1} \frac{1}{i+4} \stackrel{(63)}{\leq} \frac{C_1 L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\rho_k} \log(k+4)}{2L_{\nabla c^2}^2(k-1)(k+4)}.$$
 (65)

Using (46), (64), and the expressions of  $\rho_k$  and  $L_{\rho_k}$  in (13) and (48), we have

$$\begin{split} \|[c(x_k)]_+\| &\stackrel{(46)}{\leq} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2(F_{\rho_k}(x_k) - F_{\rho_k}^*)}{\rho_k}} \stackrel{(64)}{\leq} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2}{\rho_k} \left(\frac{C_1 L_{\rho_k}}{(k-1)(k+4)} + C_2\right)} \\ \stackrel{(13)(48)}{\equiv} \frac{2\Lambda}{(k+4)^{3/2}} + \sqrt{\frac{2C_1 (L_{\nabla f} + (k+4)^{3/2} L_{\nabla c^2})}{(k-1)(k+4)^{5/2}}} + \frac{2C_2}{(k+4)^{3/2}}, \\ &\leq \frac{2\Lambda}{(k+4)^{3/2}} + \sqrt{\frac{2C_1 L_{\nabla f}}{(k-1)(k+4)^{5/2}}} + \sqrt{\frac{2C_1 L_{\nabla c^2}}{(k-1)(k+4)}} + \sqrt{\frac{2C_2}{(k+4)^{3/2}}}, \end{split}$$

which, along with  $k-1 \ge k/2$  for all  $k \ge 2$ , implies that (14) holds. In addition, by (65), and the expressions of  $\rho_k$  and  $L_{\rho_k}$  in (13) and (48), one has

$$\mathbb{E}[F_{\rho_k}(x_k) - F_{\rho_k}^*] \stackrel{(65)}{\leq} \frac{C_1 L_{\rho_k}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\rho_k} \log(k+4)}{2L_{\nabla c^2}^2(k-1)(k+4)} \\ \stackrel{(13)(48)}{=} \frac{C_1 L_{\nabla f}}{(k-1)(k+4)} + \frac{C_1 L_{\nabla c^2}(k+4)^{1/2}}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{2L_{\nabla c^2}^2(k-1)(k+4)} + \frac{\sigma^2 L_{\nabla c^2}(k+4)^{1/2} \log(k+4)}{2L_{\nabla c^2}^2(k-1)} \\ \stackrel{(66)}{=} \frac{C_1 L_{\nabla f}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\nabla c^2}(k-1)(k+4)}{2L_{\nabla c^2}^2(k-1)(k+4)} + \frac{\sigma^2 L_{\nabla c^2}(k+4)^{1/2} \log(k+4)}{2L_{\nabla c^2}^2(k-1)} \\ \stackrel{(66)}{=} \frac{C_1 L_{\nabla f}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\nabla c^2}(k-1)(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{2L_{\nabla c^2}^2(k-1)(k+4)} \\ \stackrel{(66)}{=} \frac{C_1 L_{\nabla f}}{(k-1)(k+4)} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{2L_{\nabla c^2}^2(k-1)(k+4)} \\ \stackrel{(66)}{=} \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} \\ \stackrel{(66)}{=} \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} \\ \stackrel{(66)}{=} \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} \\ \stackrel{(66)}{=} \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} \\ \stackrel{(66)}{=} \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} \\ \stackrel{(66)}{=} \frac{\sigma^2 L_{\nabla f} \log(k+4)}{k-1} + \frac{\sigma^2 L_{\nabla$$

$$\leq 2C_1 L_{\nabla f} k^{-2} + 2\sqrt{3}C_1 L_{\nabla c^2} k^{-1/2} + 3\sigma^2 L_{\nabla f} L_{\nabla c^2}^{-2} k^{-2} \log k + 3\sqrt{3}\sigma^2 L_{\nabla c^2}^{-1} k^{-1/2} \log k, \tag{67}$$

where the last inequality is due to the fact that  $k-1 \ge k/2$ ,  $k+4 \le 3k$ , and  $\log(k+4) \le 3\log k$  for all  $k \ge 2$ . The relation (15) then follows from (67) and  $\mathbb{E}[F(x_k) - F^*] \le \mathbb{E}[F_{\rho_k}(x_k) - F^*_{\rho_k}]$  due to (45). Legally, it follows from (13), (46), and (66) that

Lastly, it follows from (13), (46), and (66) that

$$\begin{split} \mathbb{E}[\|[c(x_k)]_+\|] &\stackrel{(46)}{\leq} \frac{2\Lambda}{\rho_k} + \mathbb{E}\left[\sqrt{\frac{2F_{\rho_k}(x_k) - F_{\rho_k}^*}{\rho_k}}\right] \leq \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2\mathbb{E}\left[F_{\rho_k}(x_k) - F_{\rho_k}^*\right]}{\rho_k}} \\ &\stackrel{(13)(67)}{\leq} \frac{2\Lambda}{(k+4)^{3/2}} + \sqrt{\frac{2C_1L_{\nabla f}}{(k-1)(k+4)^{5/2}}} + \sqrt{\frac{2C_1L_{\nabla c^2}}{(k-1)(k+4)}} + \sqrt{\frac{\sigma^2L_{\nabla f}\log(k+4)}{L_{\nabla c^2}^2(k-1)(k+4)^{5/2}}} \\ &\quad + \sqrt{\frac{\sigma^2L_{\nabla c^2}\log(k+4)}{L_{\nabla c^2}^2(k-1)(k+4)}}. \\ &\leq 2\Lambda k^{-3/2} + 2(C_1L_{\nabla f})^{1/2}k^{-7/4} + 2(C_1L_{\nabla c^2})^{1/2}k^{-1} + \sigma L_{\nabla c^2}^{-1}(6L_{\nabla f})^{1/2}k^{-7/4}(\log k)^{1/2} \\ &\quad + \sqrt{6\sigma(L_{\nabla c^2})^{-1/2}k^{-1}}(\log k)^{1/2}, \end{split}$$

where the last inequality is due to the fact that  $k-1 \ge k/2$  and  $\log(k+4) \le 3 \log k$  for all  $k \ge 2$ . This together with (45) implies that (16) holds.

#### 6.2 Proof for the main results in Section 3

In this subsection, we first establish several technical lemmas and then use them together with Lemma 1 to prove Theorems 3 and 4.

Recall that  $L_{\nabla c^2}$  and  $L_{\nabla \bar{f}}$  are defined in (7) and (20), respectively. For notational convenience, we define

$$f_{\rho}(x) := \bar{f}(x) + \frac{\rho}{2} \| [c(x)]_{+} \|^{2}, \qquad L_{\rho} := L_{\nabla \bar{f}} + \rho L_{\nabla c^{2}}, \\ \ell_{f_{\rho}}(u, v) := f_{\rho}(v) + \langle \nabla f_{\rho}(v), u - v \rangle$$
(68)

for any  $\rho > 0$ . Clearly, it follows from these and Assumption 3 that  $\bar{f}$  is  $L_{\nabla \bar{f}}$ -smooth, and  $f_{\rho}$  is  $L_{\rho}$ -smooth.

The following lemma shows that  $g_t$  is an unbiased stochastic estimator of  $\nabla f_{\rho_k}(y_t)$  and also provides an upper bound on its variance.

**Lemma 5.** Suppose that Assumptions 1 and 3 hold. Let  $\{g_t\}$  and  $\{y_t\}$  be generated in the kth outer iteration of Algorithm 2 with  $\{q_i\}$  given in (27). Then for all  $1 \le t \le T_k$ , we have

$$\mathbb{E}[\delta_t | \Xi_{k,t-1}] = 0, \tag{69}$$

$$\mathbb{E}[\|\delta_t\|^2 | \Xi_{k,t-1}] \le 2L_{\nabla \bar{f}}(\bar{f}(\tilde{x}_k) - \bar{f}(y_t) - \langle \nabla \bar{f}(y_t), \tilde{x}_k - y_t \rangle), \tag{70}$$

where f is given in (18),  $L_{\nabla \bar{f}}$  is defined in (20),  $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$  with  $f_{\rho}$  given in (68), and  $\Xi_{k,t-1}$  denotes the collection of all samples that have been generated by Algorithm 2 before the t-th inner iteration of the kth outer iteration.

*Proof.* In view of (68) and the expressions of  $g_t$  and  $\delta_t$ , one can observe that

$$\delta_t = g_t - \nabla f_{\rho_k}(y_t) = (\nabla f_{i_t}(y_t) - \nabla f_{i_t}(\tilde{x}_k)) / (q_{i_t}s) + \nabla \bar{f}(\tilde{x}_k) - \nabla \bar{f}(y_t).$$

The rest proof of this lemma follows from this relation and similar arguments used in the proof of [19, Lemma 3].  $\Box$ 

The next lemma presents a well-known result for an inexact proximal gradient step applied to the problem  $\min_x \{f_{\rho_k}(x) + \psi(x)\}$ , whose proof is similar to that of [19, Lemma 5].

**Lemma 6.** Suppose that Assumptions 1 and 3 hold. Let  $\{g_t\}$ ,  $\{y_t\}$ ,  $\{z_t\}$  be generated in the kth outer iteration of Algorithm 2, and  $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$ , where  $f_{\rho_k}$  is defined in (68). Then for any  $x \in \mathcal{X}$  and  $1 \leq t \leq T_k$ , we have

$$\gamma_k \left( \ell_{f_{\rho_k}}(z_t, y_t) - \ell_{f_{\rho_k}}(x, y_t) + \psi(z_t) - \psi(x) \right) \le \frac{1}{2} \|z_{t-1} - x\|^2 - \frac{1}{2} \|z_t - x\|^2 - \frac{1}{2} \|z_{t-1} - z_t\|^2 - \gamma_k \langle \delta_t, z_t - x \rangle,$$
(71)

where  $\ell_{f_{\rho_k}}$  is defined in (68).

The following lemma establishes some recursions for  $F_{\rho_k}(x_t) - F_{\rho_k}(x)$  and  $\mathbb{E}[F_{\rho_k}(x_t) - F_{\rho_k}(x)]$  for all  $x \in \mathcal{X}$ .

**Lemma 7.** Suppose that Assumptions 1 and 3 hold. Let  $\{x_t\}$ ,  $\{y_t\}$ ,  $\{z_t\}$ ,  $\{g_t\}$ , and  $\tilde{x}_k$  be generated in the kth outer iteration of Algorithm 2 with  $\{q_i\}$  given in (27), and  $\alpha_k$ ,  $\gamma_k$ ,  $p_k$  and  $\rho_k$  satisfying

$$1 - L_{\rho_k} \alpha_k \gamma_k > 0, \tag{72}$$

$$p_k - \frac{L_{\nabla \bar{f}} \alpha_k \gamma_k}{1 - L_{\rho_k} \alpha_k \gamma_k} \ge 0 \tag{73}$$

for all  $k \ge 1$ , and let  $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$ , where  $L_{\rho_k}$  and  $f_{\rho_k}$  are given in (68). Then for all  $k \ge 1$ ,  $1 \le t \le T_k$ , and  $x \in \mathcal{X}$ , we have

$$\frac{\gamma_k}{\alpha_k} (F_{\rho_k}(x_t) - F_{\rho_k}(x)) + \frac{1}{2} \|z_t - x\|^2 \le \frac{\gamma_k}{\alpha_k} (1 - \alpha_k - p_k) (F_{\rho_k}(x_{t-1}) - F_{\rho_k}(x)) + \frac{\gamma_k p_k}{\alpha_k} (F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}(x)) + \frac{1}{2} \|z_{t-1} - x\|^2 - \gamma_k \langle \delta_t, z_t - x \rangle,$$
(74)

$$\frac{\gamma_k}{\alpha_k} \mathbb{E}[F_{\rho_k}(x_t) - F_{\rho_k}(x)] + \frac{1}{2} \mathbb{E}[\|z_t - x\|^2] \le \frac{\gamma_k}{\alpha_k} (1 - \alpha_k - p_k) \mathbb{E}[F_{\rho_k}(x_{t-1}) - F_{\rho_k}(x)] + \frac{\gamma_k p_k}{\alpha_k} \mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}(x)] + \frac{1}{2} \mathbb{E}[\|z_{t-1} - x\|^2],$$
(75)

where  $F_{\rho_k}$  is defined in (44).

*Proof.* Let  $\{x_t\}$ ,  $\{y_t\}$  and  $\{z_t\}$  be generated in the kth outer iteration of Algorithm 2. Then it follows from the expressions of  $x_t$  and  $y_t$  in Algorithm 2 that  $x_t - y_t = \alpha_k(z_t - z_{t-1})$ . By this, (71), the

 $L_{\rho_k}$ -smoothness of  $f_{\rho_k}$ , and the definitions of  $x_t$  and  $\ell_{f_{\rho_k}}$ , one has

$$\begin{aligned} f_{\rho_{k}}(x_{t}) &\leq \ell_{f_{\rho_{k}}}(x_{t}, y_{t}) + \frac{L_{\rho_{k}}}{2} \|x_{t} - y_{t}\|^{2} \\ &= (1 - \alpha_{k} - p_{k})\ell_{f_{\rho_{k}}}(x_{t-1}, y_{t}) + \alpha_{k}\ell_{f_{\rho_{k}}}(z_{t}, y_{t}) + p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) + \frac{L_{\rho_{k}}\alpha_{k}^{2}}{2} \|z_{t} - z_{t-1}\|^{2} \\ &\stackrel{(71)}{\leq} (1 - \alpha_{k} - p_{k})\ell_{f_{\rho_{k}}}(x_{t-1}, y_{t}) + \alpha_{k} \Big(\ell_{f_{\rho_{k}}}(x, y_{t}) + \psi(x) - \psi(z_{t}) + \frac{1}{2\gamma_{k}} \big(\|z_{t-1} - x\|^{2} - \|z_{t} - x\|^{2} \\ &- \|z_{t-1} - z_{t}\|^{2} \big) - \langle \delta_{t}, z_{t} - x \rangle \Big) + p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) + \frac{L_{\rho_{k}}\alpha_{k}^{2}}{2} \|z_{t} - z_{t-1}\|^{2} \\ &= (1 - \alpha_{k} - p_{k})\ell_{f_{\rho_{k}}}(x_{t-1}, y_{t}) + \alpha_{k} \Big(\ell_{f_{\rho_{k}}}(x, y_{t}) + \psi(x) - \psi(z_{t}) + \frac{1}{2\gamma_{k}} \|z_{t-1} - x\|^{2} - \frac{1}{2\gamma_{k}} \|z_{t} - x\|^{2} \Big) \\ &+ p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) - \frac{\alpha_{k}}{2\gamma_{k}}(1 - L_{\rho_{k}}\alpha_{k}\gamma_{k}) \|z_{t-1} - z_{t}\|^{2} - \alpha_{k}\langle\delta_{t}, z_{t} - x\rangle. \end{aligned}$$

$$(76)$$

Using the convexity of  $f_{\rho_k}$ , we have  $\ell_{f_{\rho_k}}(x, y_t) \leq f_{\rho_k}(x)$  for any  $x \in \mathcal{X}$ . By this, (76) and the definition of  $F_{\rho_k}$  in (19), one has

$$f_{\rho_k}(x_t) \le (1 - \alpha_k - p_k) f_{\rho_k}(x_{t-1}) + \alpha_k \Big( F_{\rho_k}(x) - \psi(z_t) + \frac{1}{2\gamma_k} \|z_{t-1} - x\|^2 - \frac{1}{2\gamma_k} \|z_t - x\|^2 \Big) \\ + p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) - \frac{\alpha_k}{2\gamma_k} (1 - L_{\rho_k} \alpha_k \gamma_k) \|z_{t-1} - z_t\|^2 - \alpha_k \langle \delta_t, z_t - x \rangle.$$

Also, by the expression of  $x_t$  and the convexity of  $\psi$ , one has

$$\psi(x_t) \le (1 - \alpha_k - p_k)\psi(x_{t-1}) + \alpha_k\psi(z_t) + p_k\psi(\tilde{x}_k).$$

Summing up these two inequalities, and using the definition of  $F_{\rho_k}$  in (19), we have

$$F_{\rho_k}(x_t) \le (1 - \alpha_k - p_k) F_{\rho_k}(x_{t-1}) + \alpha_k \Big( F_{\rho_k}(x) + \frac{1}{2\gamma_k} \|z_{t-1} - x\|^2 - \frac{1}{2\gamma_k} \|z_t - x\|^2 \Big) + p_k \ell_{f_{\rho_k}}(\tilde{x}_k, y_t) \\ - \frac{\alpha_k}{2\gamma_k} (1 - L_{\rho_k} \alpha_k \gamma_k) \|z_{t-1} - z_t\|^2 - \alpha_k \langle \delta_t, z_t - x \rangle + p_k \psi(\tilde{x}_k).$$

$$\tag{77}$$

Further, by (72), (77),  $\ell_{f_{\rho_k}}(\tilde{x}_k, y_t) \leq f_{\rho_k}(\tilde{x}_k)$ , and the definition of  $F_{\rho_k}$  in (19), one has

$$F_{\rho_k}(x_t) \le (1 - \alpha_k - p_k)F_{\rho_k}(x_{t-1}) + \alpha_k \Big(F_{\rho_k}(x) + \frac{1}{2\gamma_k} \|z_{t-1} - x\|^2 - \frac{1}{2\gamma_k} \|z_t - x\|^2\Big) + p_k F_{\rho_k}(\tilde{x}_k) - \alpha_k \langle \delta_t, z_t - x \rangle.$$

The relation (74) then follows from this inequality by rearranging terms.

We next prove that (75) holds. Indeed, observe that

$$p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) - \frac{\alpha_{k}}{2\gamma_{k}}(1 - L_{\rho_{k}}\alpha_{k}\gamma_{k})\|z_{t-1} - z_{t}\|^{2} - \alpha_{k}\langle\delta_{t}, z_{t} - x\rangle$$

$$= p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) - \frac{\alpha_{k}}{2\gamma_{k}}(1 - L_{\rho_{k}}\alpha_{k}\gamma_{k})\|z_{t-1} - z_{t}\|^{2} - \alpha_{k}\langle\delta_{t}, z_{t} - z_{t-1}\rangle - \alpha_{k}\langle\delta_{t}, z_{t-1} - x\rangle$$

$$\leq p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) + \frac{\alpha_{k}\gamma_{k}\|\delta_{t}\|^{2}}{2(1 - L_{\rho_{k}}\alpha_{k}\gamma_{k})} - \alpha_{k}\langle\delta_{t}, z_{t-1} - x\rangle, \qquad (78)$$

where the last relation follows from (72) and the Young's inequality  $a||u||^2 + b^2||v||^2/a \ge 2b\langle u, v\rangle$  for any a > 0 and b. In addition, let  $\ell_{\bar{f}}(u, v) = \bar{f}(v) + \langle \nabla \bar{f}(v), u - v \rangle$ . Notice from this and (68) that

$$\ell_{f_{\rho_k}}(u,v) = \ell_{\bar{f}}(u,v) + \frac{\rho_k}{2} \big( \|[c(v)]_+\|^2 + \langle \nabla(\|[c(v)]_+\|^2), u - v \rangle \big) \qquad \forall u, v.$$

Using this, (69), (70), (73), and the convexity of  $\overline{f}$  and  $||[c(\cdot)]_+||^2$ , we obtain that

$$p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) + \frac{\alpha_{k}\gamma_{k}\mathbb{E}[\|\delta_{t}\|^{2}|\Xi_{k,t-1}]}{2(1 - L_{\rho_{k}}\alpha_{k}\gamma_{k})} - \alpha_{k}\mathbb{E}[\langle\delta_{t}, z_{t-1} - x\rangle|\Xi_{k,t-1}]$$

$$\stackrel{(69)(70)}{\leq} p_{k}\ell_{f_{\rho_{k}}}(\tilde{x}_{k}, y_{t}) + \frac{L_{\nabla\bar{f}}\alpha_{k}\gamma_{k}}{1 - L_{\rho_{k}}\alpha_{k}\gamma_{k}}\left(\bar{f}(\tilde{x}_{k}) - \ell_{\bar{f}}(\tilde{x}_{k}, y_{t})\right)$$

$$= p_{k}\ell_{\bar{f}}(\tilde{x}_{k}, y_{t}) + \frac{L_{\nabla\bar{f}}\alpha_{k}\gamma_{k}}{1 - L_{\rho_{k}}\alpha_{k}\gamma_{k}}\left(\bar{f}(\tilde{x}_{k}) - \ell_{\bar{f}}(\tilde{x}_{k}, y_{t})\right) + \frac{p_{k}\rho_{k}}{2}\left(\|[c(y_{t})]_{+}\|^{2} + \langle\nabla(\|[c(y_{t})]_{+}\|^{2}), \tilde{x}_{k} - y_{t}\rangle\right)\right)$$

$$= \left(p_{k} - \frac{L_{\nabla\bar{f}}\alpha_{k}\gamma_{k}}{1 - L_{\rho_{k}}\alpha_{k}\gamma_{k}}\right)\ell_{\bar{f}}(\tilde{x}_{k}, y_{t}) + \frac{L_{\nabla\bar{f}}\alpha_{k}\gamma_{k}}{1 - L_{\rho_{k}}\alpha_{k}\gamma_{k}}\bar{f}(\tilde{x}_{k}) + \frac{p_{k}\rho_{k}}{2}\left(\|[c(y_{t})]_{+}\|^{2} + \langle\nabla(\|[c(y_{t})]_{+}\|^{2}), \tilde{x}_{k} - y_{t}\rangle\right)\right)$$

$$\leq p_{k}\bar{f}(\tilde{x}_{k}) + \frac{p_{k}\rho_{k}}{2}\|[c(\tilde{x}_{k})]_{+}\|^{2} = p_{k}f_{\rho_{k}}(\tilde{x}_{k}), \qquad(79)$$

where the last inequality follows from (73) and the convexity of  $\bar{f}$  and  $||[c(\cdot)]_+||^2$ . It then follows from (78) and (79) that

$$\mathbb{E}\Big[p_k\ell_{f_{\rho_k}}(\tilde{x}_k, y_t) - \frac{\alpha_k}{2\gamma_k}(1 - L_{\rho_k}\alpha_k\gamma_k) \|z_{t-1} - z_t\|^2 - \alpha_k\langle\delta_t, z_t - x\rangle\Big] \\
\stackrel{(78)}{\leq} \mathbb{E}\Big[p_k\ell_{f_{\rho_k}}(\tilde{x}_k, y_t) + \frac{\alpha_k\gamma_k\|\delta_t\|^2}{2(1 - L_{\rho_k}\alpha_k\gamma_k)} - \alpha_k\langle\delta_t, z_{t-1} - x\rangle\Big] \stackrel{(79)}{\leq} p_k\mathbb{E}[f_{\rho_k}(\tilde{x}_k)].$$

Using this inequality, the definition of  $F_{\rho_k}$  in (19), and taking expectation on both sides of (77) lead to

$$\mathbb{E}[F_{\rho_k}(x_t)] \le (1 - \alpha_k - p_k) \mathbb{E}[F_{\rho_k}(x_{t-1})] + \alpha_k \Big(F_{\rho_k}(x) + \frac{1}{2\gamma_k} \mathbb{E}[\|z_{t-1} - x\|^2] - \frac{1}{2\gamma_k} \mathbb{E}[\|z_t - x\|^2]\Big) + p_k \mathbb{E}[F_{\rho_k}(\tilde{x}_k)].$$

The relation (75) then follows from this inequality by rearranging the terms.

We next derive upper bounds for  $F_{\rho}(x_k) - F_{\rho}^*$  and  $\mathbb{E}[F_{\rho}(x_k) - F_{\rho}^*]$  under the setting where constant penalty parameter  $\rho_k \equiv \rho$  is used in Algorithm 2 for some  $\rho > 0$ .

**Lemma 8.** Suppose that Assumptions 1 and 3 hold, and that  $\{\tilde{x}_k\}$  is generated by Algorithm 2 with  $\{q_i\}, \{\theta_t\}, \{\gamma_k\}, \{p_k\}, \{\alpha_k\}$  and  $\{T_k\}$  given in (21), and  $\rho_k \equiv \rho$  for some  $\rho > 0$ . Let  $k_0 = \lfloor \log_2 s \rfloor + 1$ . Then for all  $K > k_0$ , we have

$$F_{\rho}(\tilde{x}_{K}) - F_{\rho}^{*} \leq \frac{16(C_{3} + \rho C_{4})}{s(K - k_{0} + 3)^{2}} + 16L_{\nabla \bar{f}}D_{\mathcal{X}}^{2},$$
(80)

$$\mathbb{E}[F_{\rho}(\tilde{x}_K) - F_{\rho}^*] \le \frac{16(C_3 + \rho C_4)}{s(K - k_0 + 3)^2},\tag{81}$$

where  $D_{\mathcal{X}}$  is given in Assumption 1,  $L_{\nabla \bar{f}}$  is defined in (20), and  $C_3$ ,  $C_4$ ,  $F_{\rho}$ , and  $F_{\rho}^*$  are defined in (26) and (44), respectively.

*Proof.* Let  $\{g_t\}$  and  $\{y_t\}$  be generated in the kth outer iteration of Algorithm 2, and  $\delta_t = g_t - \nabla f_{\rho_k}(y_t)$ . Notice from (27) that  $q_i = L_i/(\sum_{j=1}^s L_j)$ . By these, (20), the expression of  $g_t$ ,  $L_i$ -smoothness of  $f_i$ ,  $L_{\nabla \bar{f}}$ -smoothness of f, and Assumption 1, one has

$$\begin{aligned} \|\delta_t\| &= \left\| \frac{\nabla f_{i_t}(y_t) - \nabla f_{i_t}(\tilde{x}_k)}{q_{i_t}s} + \nabla f(\tilde{x}_k) - \nabla f(y_t) \right\| \leq \frac{L_{i_t} \|y_t - \tilde{x}_k\|}{q_{i_t}s} + L_{\nabla \bar{f}} \|\tilde{x}_k - y_t\| \\ &\leq \left(\frac{L_{i_t}}{q_{i_t}s} + L_{\nabla \bar{f}}\right) D_{\mathcal{X}} \stackrel{(20)}{=} 2L_{\nabla \bar{f}} D_{\mathcal{X}}. \end{aligned}$$

This inequality and Assumption 1 imply that

$$\langle \delta_t, z_t - x \rangle \le \|\delta_t\| \|z_t - x\| \le 2L_{\nabla \bar{f}} D_{\mathcal{X}}^2 \qquad \forall x \in \mathcal{X}.$$
(82)

Let  $L_{\rho}$  be defined in (68). Notice from (21) and (68) that  $1/(3L_{\rho_k}\alpha_k) = \gamma_k \leq 1/(3L_{\nabla \bar{f}}\alpha_k)$ , which implies that  $L_{\rho_k}\alpha_k\gamma_k = 1/3$  and  $L_{\nabla \bar{f}}\alpha_k\gamma_k \leq 1/3$ . By these, one can verify that the assumption of Lemma 7 holds for  $\{q_i\}$ ,  $\{\alpha_k\}$ ,  $\{\gamma_k\}$  and  $\{p_k\}$  given in (21). Let  $x_{\rho}^* \in \arg\min_{x \in \mathcal{X}} F_{\rho}(x)$ . It then follows from (74) with  $x = x_{\rho}^*$  and (82) that

$$\frac{\gamma_k}{\alpha_k} (F_{\rho}(x_t) - F_{\rho}^*) \le \frac{\gamma_k}{\alpha_k} (1 - \alpha_k - p_k) (F_{\rho}(x_{t-1}) - F_{\rho}^*) + \frac{\gamma_k p_k}{\alpha_k} (F_{\rho}(\tilde{x}_k) - F_{\rho}^*) + \frac{1}{2} (\|z_{t-1} - x_{\rho}^*\|^2 - \|z_t - x_{\rho}^*\|^2) + 2\gamma_k L_{\nabla \bar{f}} D_{\mathcal{X}}^2.$$
(83)

Summing up (83) from t = 1 to  $T_k$ , and using the fact that  $x_0 = \tilde{x}_k$ ,  $z_0 = \tilde{z}_k$ ,  $z_{T_k} = \tilde{z}_{k+1}$ , we have

$$\sum_{t=1}^{T_{k}} \frac{\gamma_{k}}{\alpha_{k}} (F_{\rho}(x_{t}) - F_{\rho}^{*}) \leq \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho}(x_{0}) - F_{\rho}^{*}) + \sum_{t=2}^{T_{k}} \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho}(x_{t-1}) - F_{\rho}^{*}) + \frac{T_{k} \gamma_{k} p_{k}}{\alpha_{k}} (F_{\rho}(\tilde{x}_{k}) - F_{\rho}^{*}) + \frac{1}{2} (||z_{0} - x_{\rho}^{*}||^{2} - ||z_{T_{k}} - x_{\rho}^{*}||^{2}) + 2\gamma_{k} T_{k} L_{\nabla \bar{f}} D_{\mathcal{X}}^{2} = \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho}(\tilde{x}_{k}) - F_{\rho}^{*}) + \sum_{t=1}^{T_{k}-1} \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho}(x_{t}) - F_{\rho}^{*}) + \frac{T_{k} \gamma_{k} p_{k}}{\alpha_{k}} (F_{\rho}(\tilde{x}_{k}) - F_{\rho}^{*}) + \frac{1}{2} (||\tilde{z}_{k} - x_{\rho}^{*}||^{2} - ||\tilde{z}_{k+1} - x_{\rho}^{*}||^{2}) + 2\gamma_{k} T_{k} L_{\nabla \bar{f}} D_{\mathcal{X}}^{2}.$$

$$(84)$$

Let  $\mathcal{L}_k$  and  $\mathcal{R}_k$  be defined as

$$\mathcal{L}_k := \frac{\gamma_k}{\alpha_k} + (T_k - 1) \frac{\gamma_k (\alpha_k + p_k)}{\alpha_k}, \qquad \mathcal{R}_k := \frac{\gamma_k}{\alpha_k} (1 - \alpha_k) + (T_k - 1) \frac{\gamma_k p_k}{\alpha_k}$$
(85)

with the associated  $\alpha_k$ ,  $\gamma_k$ ,  $p_k$  and  $T_k$  given in (21). Notice from (21), the definition of  $\mathcal{L}_k$ , and Algorithm 2 that  $\mathcal{L}_k = \sum_{t=1}^{T_k} \theta_t$  and  $\tilde{x}_{k+1} = \sum_{t=1}^{T_k} (\theta_t x_t) / \sum_{t=1}^{T_k} \theta_t$ . Using these, (84), and the convexity of  $F_{\rho}$ , we have

$$\mathcal{L}_{k}(F_{\rho}(\tilde{x}_{k+1}) - F_{\rho}^{*}) = \left(\sum_{t=1}^{T_{k}} \theta_{t}\right)(F_{\rho}(\tilde{x}_{k+1}) - F_{\rho}^{*}) \leq \sum_{t=1}^{T_{k}} \theta_{t}(F_{\rho}(x_{t}) - F_{\rho}^{*})$$

$$\stackrel{(21)}{=} \sum_{t=1}^{T_{k}} \frac{\gamma_{k}}{\alpha_{k}}(F_{\rho}(x_{t}) - F_{\rho}^{*}) - \sum_{t=1}^{T_{k}-1} \frac{\gamma_{k}}{\alpha_{k}}(1 - \alpha_{k} - p_{k})(F_{\rho}(x_{t}) - F_{\rho}^{*})$$

$$\stackrel{(84)}{\leq} \left(\frac{\gamma_{k}}{\alpha_{k}}(1 - \alpha_{k}) + (T_{k} - 1)\frac{\gamma_{k}p_{k}}{\alpha_{k}}\right)(F_{\rho}(\tilde{x}_{k}) - F_{\rho}^{*}) + \frac{1}{2}\left(\|\tilde{z}_{k} - x_{\rho}^{*}\|^{2} - \|\tilde{z}_{k+1} - x_{\rho}^{*}\|^{2}\right) + 2\gamma_{k}T_{k}L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}$$

$$= \mathcal{R}_{k}(F_{\rho}(\tilde{x}_{k}) - F_{\rho}^{*}) + \frac{1}{2}\left(\|\tilde{z}_{k} - x_{\rho}^{*}\|^{2} - \|\tilde{z}_{k+1} - x_{\rho}^{*}\|^{2}\right) + 2\gamma_{k}T_{k}L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}.$$
(86)

Next claim that  $\mathcal{L}_k - \mathcal{R}_{k+1} \ge 0$  for all  $k \ge 1$ . Indeed, notice from (21) that if  $1 \le k < k_0$ ,  $\alpha_{k+1} = \alpha_k = 1/2$ ,  $p_k = 1/2$ ,  $\gamma_{k+1} = \gamma_k$ , and  $T_{k+1} = 2T_k$ . Using these and (85), we have that for all  $1 \le k < k_0$ ,

$$\mathcal{L}_{k} - \mathcal{R}_{k+1} \stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1) \frac{\gamma_{k}(\alpha_{k} + p_{k})}{\alpha_{k}} - \frac{\gamma_{k+1}}{\alpha_{k+1}} (1 - \alpha_{k+1}) + (T_{k+1} - 1) \frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\ = \frac{\gamma_{k}}{\alpha_{k}} (1 + (T_{k} - 1)(\alpha_{k} + p_{k}) - (1 - \alpha_{k}) - (2T_{k} - 1)p_{k}) = 0,$$

and hence the above claim holds for all  $1 \leq k < k_0$ . On the other hand, if  $k \geq k_0$ , notice from (21), (68),  $k_0 \geq 1$ , and  $\rho_k \equiv \rho$  that  $T_k = T_{k+1} = T_{k_0} \geq 1$ ,  $\alpha_k = 2/(k - k_0 + 4)$ ,  $p_k = 1/2$ , and  $\gamma_k = 1/(3L_\rho\alpha_k) = (k - k_0 + 4)/(6L_\rho)$ . Using these and (85), we obtain that for all  $k \geq k_0$ ,

$$\begin{aligned} \mathcal{L}_{k} - \mathcal{R}_{k+1} &\stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1) \frac{\gamma_{k}(\alpha_{k} + p_{k})}{\alpha_{k}} - \frac{\gamma_{k+1}}{\alpha_{k+1}} (1 - \alpha_{k+1}) - (T_{k+1} - 1) \frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\ &= \frac{\gamma_{k}}{\alpha_{k}} + (T_{k_{0}} - 1) \frac{\gamma_{k}(\alpha_{k} + p_{k})}{\alpha_{k}} - \frac{\gamma_{k+1}}{\alpha_{k+1}} (1 - \alpha_{k+1}) - (T_{k_{0}} - 1) \frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\ &= \frac{\gamma_{k}}{\alpha_{k}} - \frac{\gamma_{k+1}}{\alpha_{k+1}} (1 - \alpha_{k+1}) + (T_{k_{0}} - 1) \left( \frac{\gamma_{k}(\alpha_{k} + p_{k})}{\alpha_{k}} - \frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \right) \\ &= \frac{(k - k_{0} + 4)^{2}}{12L_{\rho}} - \frac{(k - k_{0} + 5)^{2}}{12L_{\rho}} \left( 1 - \frac{2}{k - k_{0} + 5} \right) \\ &+ (T_{k_{0}} - 1) \left[ \frac{(k - k_{0} + 4)^{2}}{12L_{\rho}} \left( \frac{2}{k - k_{0} + 4} + \frac{1}{2} \right) - \frac{(k - k_{0} + 5)^{2}}{24L_{\rho}} \right] \\ &= \frac{(K - k_{0} + 4)^{2} - (k - k_{0} + 5)^{2} + 2(k - k_{0} + 5)}{12L_{\rho}} \\ &+ \frac{(T_{k_{0}} - 1) \left[ 4(k - k_{0} + 4) + (k - k_{0} + 4)^{2} - (k - k_{0} + 5)^{2} \right]}{24L_{\rho}} \\ &= \frac{1}{12L_{\rho}} + \frac{(T_{k_{0}} - 1)(2(k - k_{0}) + 7)}{24L_{\rho}} \ge 0,
\end{aligned}$$

where the second equality is due to  $T_k = T_{k+1} = T_{k_0}$ , and the fourth equality follows from  $\alpha_k = 2/(k-k_0+4)$ ,  $p_k = 1/2$  and  $\gamma_k = (k-k_0+4)/(6L_{\rho})$ . Hence, the above claim also holds for all  $k \ge k_0$ . Summing up (86) from k = 1 to K-1 and using the fact that  $F_{\epsilon}(x) - F^* \ge 0$  for any  $x \in \mathcal{X}$ .

Summing up (86) from k = 1 to K - 1, and using the fact that  $F_{\rho}(x) - F_{\rho}^* \ge 0$  for any  $x \in \mathcal{X}$ ,  $\mathcal{L}_k - \mathcal{R}_{k+1} \ge 0$  for all  $k \ge 1$ ,  $\tilde{x}_1 = \tilde{z}_1 = x^0$ , and  $\mathcal{R}_1 = 2/(3L_{\rho})$ , we have

$$\mathcal{L}_{K-1}(F_{\rho}(\tilde{x}_{K}) - F_{\rho}^{*}) \leq \sum_{k=1}^{K-2} (\mathcal{R}_{k+1} - \mathcal{L}_{k})(F_{\rho}(\tilde{x}_{k}) - F_{\rho}^{*}) + \mathcal{R}_{1}(F_{\rho}(\tilde{x}_{1}) - F_{\rho}^{*}) + \frac{1}{2} \|\tilde{z}_{1} - x_{\rho}^{*}\|^{2} + 2L_{\nabla\bar{f}}D_{\mathcal{X}}^{2} \sum_{k=1}^{K-1} \gamma_{k}T_{k}$$

$$\leq \frac{2}{3L_{\rho}}(F_{\rho}(x^{0}) - F_{\rho}^{*}) + \frac{1}{2} \|x^{0} - x_{\rho}^{*}\|^{2} + 2L_{\nabla\bar{f}}D_{\mathcal{X}}^{2} \sum_{k=1}^{K-1} \gamma_{k}T_{k}.$$
(87)

In addition, recall that  $k_0 = \lfloor \log_2 s \rfloor + 1$  and  $T_{k_0} = 2^{k_0 - 1}$ , which imply that  $s/2 \leq T_{k_0} \leq s$ . By this,  $\rho_k \equiv \rho$ , (21), (68), and (85), one has that for all  $K > k_0$ ,

$$\begin{aligned} \mathcal{L}_{K-1} \stackrel{(85)}{=} \frac{\gamma_{K-1}}{\alpha_{K-1}} \left( 1 + (T_{K-1} - 1)(\alpha_{K-1} + p_{K-1}) \right) \\ \stackrel{(21)(68)}{=} \frac{1}{3L_{\rho}\alpha_{K-1}^{2}} \left( 1 + (T_{K-1} - 1)\left(\alpha_{K-1} + \frac{1}{2}\right) \right) &= \frac{T_{K-1} - 1}{3L_{\rho}\alpha_{K-1}} + \frac{T_{K-1} + 1}{6L_{\rho}\alpha_{K-1}^{2}} \\ \stackrel{(21)}{=} \frac{(T_{k_{0}} - 1)(K - k_{0} + 3)}{6L_{\rho}} + \frac{(T_{k_{0}} + 1)(K - k_{0} + 3)^{2}}{24L_{\rho}} \\ &\geq \frac{(s - 2)(K - k_{0} + 3)}{12L_{\rho}} + \frac{(s + 2)(K - k_{0} + 3)^{2}}{48L_{\rho}} \\ &= \frac{(s - 2)(K - k_{0} + 3)}{12L_{\rho}} + \frac{2(K - k_{0} + 3)^{2}}{48L_{\rho}} + \frac{s(K - k_{0} + 3)^{2}}{48L_{\rho}} \\ &\geq \frac{(s - 2)(K - k_{0} + 3)}{12L_{\rho}} + \frac{8(K - k_{0} + 3)}{48L_{\rho}} + \frac{s(K - k_{0} + 3)^{2}}{48L_{\rho}} \end{aligned}$$

$$\begin{split} &= \frac{s(K-k_0+3)}{12L_{\rho}} + \frac{s(K-k_0+3)^2}{48L_{\rho}} \ge \frac{s(K-k_0+3)^2}{48L_{\rho}}, \\ &\sum_{k=1}^{K-1} \gamma_k T_k = \sum_{k=1}^{k_0-1} \gamma_k T_k + \sum_{k=k_0}^{K-1} \gamma_k T_k \stackrel{(21)(68)}{=} \frac{2}{3L_{\rho}} \sum_{k=1}^{k_0-1} 2^{k-1} + \frac{T_{k_0}}{3L_{\rho}} \sum_{k=k_0}^{K-1} \frac{k-k_0+4}{2} \\ &\le \frac{2^{k_0}}{3L_{\rho}} + \frac{T_{k_0}}{3L_{\rho}} \frac{(K-k_0+7)(K-k_0)}{4} = \frac{T_{k_0} \left( (K-k_0)^2 + 7(K-k_0) + 8 \right)}{12L_{\rho}} \\ &\le \frac{T_{k_0} \left( 2(K-k_0)^2 + 12(K-k_0) + 18 \right)}{12L_{\rho}} \le \frac{s(K-k_0+3)^2}{6L_{\rho}}. \end{split}$$

It then follows from the above two relations, (44), (68), (87), and Assumption 1(i) that

$$\begin{aligned} F_{\rho}(\tilde{x}_{K}) - F_{\rho}^{*} &\leq \frac{16}{s(K - k_{0} + 3)^{2}} \Big( 2(F_{\rho}(x^{0}) - F_{\rho}^{*}) + \frac{3L_{\rho}}{2} \|x^{0} - x_{\rho}^{*}\|^{2} \Big) + 16L_{\nabla\bar{f}}D_{\mathcal{X}}^{2} \\ &\stackrel{(44)}{\leq} \frac{16}{s(K - k_{0} + 3)^{2}} \Big( 2(F(x^{0}) - F(x_{\rho}^{*})) + \rho \|[c(x^{0})]_{+}\|^{2} + \frac{3L_{\rho}}{2} \|x^{0} - x_{\rho}^{*}\|^{2} \Big) + 16L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}, \\ &\leq \frac{16}{s(K - k_{0} + 3)^{2}} \Big( 2D_{F} + \rho \|[c(x^{0})]_{+}\|^{2} + \frac{3(L_{\nabla\bar{f}} + \rho L_{\nabla c^{2}})D_{\mathcal{X}}^{2}}{2} \Big) + 16L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}, \end{aligned}$$

which together with (26) implies that (80) holds. In addition, the proof of (81) follows from (75) and similar arguments as above, and is thus omitted.  $\Box$ 

We are now ready to prove Theorem 3.

**Proof of Theorem 3.** We first prove statement (i) of Theorem 3, where  $\rho = s^{2/3}K^{4/3}$ . Using (46) and (80), we have

$$\|[c(\tilde{x}_{K})]_{+}\| \stackrel{(46)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{2(F_{\rho}(\tilde{x}_{K}) - F_{\rho}^{*})}{\rho}} \stackrel{(80)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{32(C_{3} + \rho C_{4})}{\rho s(K - k_{0} + 3)^{2}}} + \frac{32L_{\nabla \bar{f}}D_{\mathcal{X}}^{2}}{\rho}$$
$$\leq \frac{2\Lambda}{\rho} + \sqrt{\frac{32C_{3}}{\rho s(K - k_{0} + 3)^{2}}} + \sqrt{\frac{32C_{4}}{s(K - k_{0} + 3)^{2}}} + \sqrt{\frac{32L_{\nabla \bar{f}}D_{\mathcal{X}}^{2}}{\rho}}.$$
(88)

Also, observe from (45) that  $\mathbb{E}[F(\tilde{x}_K) - F^*] \leq \mathbb{E}[F_{\rho}(\tilde{x}_K) - F^*_{\rho}]$ . Using these, (81),  $\rho = s^{2/3}K^{4/3}$ , and  $K \geq 2(k_0 - 3)$ , we see that (22) and (24) hold. In addition, by (46) and (81), one has

$$\mathbb{E}[\|[c(\tilde{x}_{K})]_{+}\|] \stackrel{(46)}{\leq} \frac{2\Lambda}{\rho} + \mathbb{E}\left[\sqrt{\frac{2(F_{\rho}(\tilde{x}_{K}) - F_{\rho}^{*})}{\rho}}\right] \leq \frac{2\Lambda}{\rho} + \sqrt{\frac{2\mathbb{E}[F_{\rho}(\tilde{x}_{K}) - F_{\rho}^{*}]}{\rho}} \\ \stackrel{(81)}{\leq} \frac{2\Lambda}{\rho} + \sqrt{\frac{32(C_{3} + \rho C_{4})}{\rho s(K - k_{0} + 3)^{2}}} \leq \frac{2\Lambda}{\rho} + \sqrt{\frac{32C_{3}}{\rho s(K - k_{0} + 3)^{2}}} + \sqrt{\frac{32C_{4}}{s(K - k_{0} + 3)^{2}}}. \quad (89)$$

Also, notice from (45) that  $\mathbb{E}[F(\tilde{x}_K) - F^*] \ge -\Lambda \mathbb{E}[||[c(\tilde{x}_K)]_+||]]$ . By these,  $\rho = s^{2/3}K^{4/3}$ , and  $K \ge 2(k_0 - 3)$ , one can conclude that (23) and (25) hold.

The proof of statement (ii) of Theorem 3 follows from  $\rho = \sqrt{s}K$  and similar arguments as above, and is thus omitted.

In the remainder of this subsection, we provide a proof of Theorem 4. Before proceeding, we establish several technical lemmas below. The following lemma provides a relationship between  $\mathcal{L}_k$  and  $\mathcal{R}_{k+1}$ .

**Lemma 9.** Let  $\mathcal{L}_k$  and  $\mathcal{R}_k$  be defined in (85) with  $\{\alpha_k\}$ ,  $\{\gamma_k\}$ ,  $\{p_k\}$  given in (27), and  $k_0$ ,  $\{\rho_k\}$ ,  $\{T_k\}$  given in (28) or (34). Then we have

$$\mathcal{L}_k\left(1 - \frac{2(\rho_{k+1} - \rho_k)}{\rho_{k+1}}\right) \ge \mathcal{R}_{k+1} \qquad \forall k \ge 1.$$
(90)

*Proof.* We divide the proof into three separate cases:  $1 \le k < k_0$ ,  $k = k_0$ , and  $k > k_0$ .

Case 1)  $1 \le k < k_0$ . Fix any  $1 \le k < k_0$ . Claim that  $T_k \ge 1$  and  $T_{k+1} \le 2T_k$  hold for  $\{T_k\}$  given in (28) or (34). Indeed, if  $\{T_k\}$  is given in (28), one has  $T_k = \lceil 2^{3(k-1)/4} \rceil$ , which implies that  $T_k \ge 1$  and

$$T_{k+1} = \lceil 2^{3(k+1-1)/4} \rceil = \lceil 2^{3/4} \times 2^{3(k-1)/4} \rceil \le \lceil 2^{3/4} \rceil \times \lceil 2^{3(k-1)/4} \rceil \le 2T_k$$

In addition, if  $\{T_k\}$  is given in (34), one has  $T_k = 2^{k-1}$ , yielding  $T_k \ge 1$  and  $T_{k+1} = 2T_k$ . Hence, the claim holds as desired. Observe from (27) that  $\alpha_{k+1} = \alpha_k$  and  $\rho_{k+1} \ge \rho_k$ , which along with (68) imply that  $\gamma_{k+1} \le \gamma_k$ . Also, notice from (27) that  $\alpha_{k+1} = \alpha_k = 6/7$ ,  $p_{k+1} = p_k = 1/7$ ,  $\rho_{k+1} = \sqrt{2}\rho_k$ . Using these, (85),  $T_k \ge 1$ ,  $T_{k+1} \le 2T_k$ , and  $\gamma_{k+1} \le \gamma_k$ , we obtain that

$$\begin{aligned} \mathcal{L}_{k} - \mathcal{R}_{k+1} &\stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1) \frac{\gamma_{k}(\alpha_{k} + p_{k})}{\alpha_{k}} - \frac{\gamma_{k+1}}{\alpha_{k+1}} (1 - \alpha_{k+1}) + (T_{k+1} - 1) \frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\ &\geq \frac{\gamma_{k}}{\alpha_{k}} (1 + (T_{k} - 1)(\alpha_{k} + p_{k})) - \frac{\gamma_{k+1}}{\alpha_{k}} ((1 - \alpha_{k}) + (2T_{k} - 1)p_{k}) \\ &\geq \frac{\gamma_{k}}{\alpha_{k}} (1 + (T_{k} - 1)(\alpha_{k} + p_{k}) - (1 - \alpha_{k}) - (2T_{k} - 1)p_{k}) \\ &= \frac{5\gamma_{k}T_{k}}{7\alpha_{k}} = \frac{5\mathcal{L}_{k}}{7} \geq \frac{2(\rho_{k+1} - \rho_{k})}{\rho_{k+1}} \mathcal{L}_{k}, \end{aligned}$$

where the first inequality is due to  $\alpha_{k+1} = \alpha_k$  and  $T_{k+1} \leq 2T_k$ , the second inequality follows from  $\gamma_{k+1} \leq \gamma_k$ ,  $0 < \alpha_k < 1$  and  $T_k \geq 1$ , the second and third equalities are due to  $\alpha_k = 6/7$ ,  $p_k = 1/7$ , and (85), and the last inequality follows from  $\rho_{k+1} = \sqrt{2}\rho_k$ . Hence, (90) holds for all  $1 \leq k < k_0$ .

Case 2)  $k = k_0$ . Claim that  $0 < (\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \le 1/3$  holds for  $k_0, \rho_{k_0}$ , and  $\rho_{k_0+1}$  given in (28) or (34). Indeed, if they are given in (28), one can observe that  $k_0 = \lfloor (4/3) \log_2 s \rfloor + 1$ ,  $\rho_{k_0} = 2^{k_0/2}$ , and  $\rho_{k_0+1} = 3s^{2/3}/2$ . Also, notice that  $s^{2/3} \le 2^{(\lfloor (4/3) \log_2 s \rfloor + 1)/2} \le \sqrt{2}s^{2/3}$  and hence  $s^{2/3} \le \rho_{k_0} \le \sqrt{2}s^{2/3}$ . This together with  $\rho_{k_0+1} = 3s^{2/3}/2$  implies that

$$0 < (\rho_{k_0+1} - \rho_{k_0}) / \rho_{k_0+1} \le (3s^{2/3}/2 - s^{2/3}) / (3s^{2/3}/2) = 1/3.$$

In addition, if  $k_0$ ,  $\rho_{k_0}$ , and  $\rho_{k_0+1}$  are given in (34), we observe that  $k_0 = \lfloor \log_2 s \rfloor + 1$ ,  $\rho_{k_0} = 2^{k_0/2}$ , and  $\rho_{k_0+1} = 3\sqrt{s}/2$ . Notice that  $\sqrt{s} \le 2^{(\lfloor \log_2 s \rfloor + 1)/2} \le \sqrt{2s}$  and hence  $\sqrt{s} \le \rho_{k_0} \le \sqrt{2s}$ . By this and  $\rho_{k_0+1} = 3\sqrt{s}/2$ , one has

$$0 < (\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \le (3\sqrt{s}/2 - \sqrt{s})/(3\sqrt{s}/2) = 1/3.$$

Hence, the above claim holds as desired. Notice from this claim, (27), and (68) that  $L_{\rho_{k_0+1}} \ge L_{\rho_{k_0}}$ ,  $\alpha_{k_0} = 6/7$ , and  $\alpha_{k_0+1} = 6/8$ . By these and (27), one has

$$\gamma_{k_0+1}/\alpha_{k_0+1} = 1/(8L_{\rho_{k_0+1}}\alpha_{k_0+1}^2) \le 1/(8L_{\rho_{k_0}}\alpha_{k_0+1}^2) = 64/(392L_{\rho_{k_0}}\alpha_{k_0}^2) = 64\gamma_{k_0}/(49\alpha_{k_0}).$$
(91)

Notice from (27), (28), (34), and (85) that  $\alpha_{k_0+1} \leq 1$ ,  $\alpha_{k_0} = 6/7$ ,  $p_{k_0+1} = p_{k_0} = 1/7$ ,  $T_{k_0+1} = T_{k_0} \geq 1$ ,

and  $\mathcal{L}_{k_0} = \gamma_{k_0} T_{k_0} / \alpha_{k_0}$ . Using these, (91), and  $(\rho_{k_0+1} - \rho_{k_0}) / \rho_{k_0+1} \leq 1/3$ , we obtain that

$$\begin{aligned} \mathcal{L}_{k_{0}} - \mathcal{R}_{k_{0}+1} \stackrel{(85)}{=} \frac{\gamma_{k_{0}}}{\alpha_{k_{0}}} + (T_{k_{0}} - 1) \frac{\gamma_{k_{0}}(\alpha_{k_{0}} + p_{k_{0}})}{\alpha_{k_{0}}} - \frac{\gamma_{k_{0}+1}}{\alpha_{k_{0}+1}} (1 - \alpha_{k_{0}+1}) - (T_{k_{0}+1} - 1) \frac{\gamma_{k_{0}+1}p_{k_{0}+1}}{\alpha_{k_{0}+1}} \\ &= \frac{\gamma_{k_{0}}}{\alpha_{k_{0}}} \Big( 1 + (T_{k_{0}} - 1)(\alpha_{k_{0}} + p_{k_{0}}) \Big) - \frac{\gamma_{k_{0}+1}}{\alpha_{k_{0}+1}} \Big( (1 - \alpha_{k_{0}+1}) + (T_{k_{0}+1} - 1)p_{k_{0}+1}) \Big) \\ \stackrel{(91)}{\geq} \frac{\gamma_{k_{0}}}{\alpha_{k_{0}}} \Big( 1 + (T_{k_{0}} - 1)(\alpha_{k_{0}} + p_{k_{0}}) - \frac{64}{49} \Big( (1 - \alpha_{k_{0}+1}) + (T_{k_{0}+1} - 1)p_{k_{0}+1}) \Big) \Big) \\ &= \frac{\gamma_{k_{0}}}{\alpha_{k_{0}}} \Big( \Big( 1 - \frac{64}{343} \Big) T_{k_{0}} - \frac{16}{49} + \frac{64}{343} \Big) \\ &\geq \frac{33\gamma_{k_{0}}T_{k_{0}}}{49\alpha_{k_{0}}} \geq \frac{2\gamma_{k_{0}}T_{k_{0}}}{3\alpha_{k_{0}}} \geq \frac{2(\rho_{k_{0}+1} - \rho_{k_{0}})}{\rho_{k_{0}+1}} \mathcal{L}_{k_{0}}, \end{aligned}$$

where the first inequality follows from (91),  $\alpha_{k_0+1} \leq 1$ , and  $T_{k_0+1} \geq 1$ , the third equality is due to  $\alpha_{k_0} = 6/7$ ,  $p_{k_0+1} = p_{k_0} = 1/7$ , and  $T_{k_0+1} = T_{k_0}$ , the second inequality follows from  $T_{k_0} \geq 1$  and -16/49 + 64/343 < 0, and the last inequality is due to  $(\rho_{k_0+1} - \rho_{k_0})/\rho_{k_0+1} \leq 1/3$  and  $\mathcal{L}_{k_0} = \gamma_{k_0}T_{k_0}/\alpha_{k_0}$ . Hence, (90) also holds for  $k = k_0$ .

Case 3)  $k > k_0$ . Fix any  $k > k_0$ . Claim that

$$\rho_{k+1} \ge \rho_k, \quad L_{\rho_{k+1}} \le 6L_{\rho_k}/5,$$
(92)

$$\rho_{k+1}^{-1}(\rho_{k+1} - \rho_k) \le 4(k - k_0 + 8)^{-1}/3 \tag{93}$$

hold for  $\{\rho_k\}$  given in (28) or (34). Indeed, suppose that  $\{\rho_k\}$  is given in (28). By this and  $k > k_0$ , one has  $\rho_k = 3s^{2/3}(k - k_0 + 7)^{4/3}/32$ . Using this,  $k > k_0$ , and (68), we obtain that  $\rho_{k+1} \ge \rho_k$ , and

$$L_{\rho_{k+1}} = L_{\nabla \bar{f}} + \rho_{k+1} L_{\nabla c^2} \le \rho_{k+1} \rho_k^{-1} L_{\nabla \bar{f}} + \rho_{k+1} L_{\nabla c^2} = \rho_{k+1} \rho_k^{-1} L_{\rho_k}$$
  
$$\le \left(\frac{k - k_0 + 8}{k - k_0 + 7}\right)^{4/3} L_{\rho_k} \le \left(\frac{8}{7}\right)^{4/3} L_{\rho_k} \le \frac{6L_{\rho_k}}{5},$$
  
$$\rho_{k+1} - \rho_k = 3s^{2/3} ((k - k_0 + 8)^{4/3} - (k - k_0 + 7)^{4/3})/32 \le s^{2/3} (k - k_0 + 8)^{1/3}/8,$$
  
$$= 4\rho_{k+1} (k - k_0 + 8)^{-1}/3,$$

where the last inequality is due to the convexity of the function  $\tau^{4/3}$ . Hence, (92) and (93) hold for  $\{\rho_k\}$  given in (28). In addition, if  $\{\rho_k\}$  is given in (34), one has  $\rho_k = 3\sqrt{s(k-k_0+7)/16}$  due to  $k > k_0$ . By this and similar arguments as above, one can show that (92) and (93) also hold for  $\{\rho_k\}$  given in (34). Hence, the above claim holds as desired.

Further, by (68) and  $\rho_{k+1} \ge \rho_k$ , one can see that  $L_{\rho_k} \le L_{\rho_{k+1}}$ . In addition, it follows from (27), (28), (34), (68), and  $k > k_0$  that  $\alpha_k = 6/(k-k_0+7)$ ,  $\alpha_{k+1} = 6/(k-k_0+8)$ ,  $p_k = p_{k+1} = 1/7$ ,  $\gamma_k = 1/(8L_{\rho_k}\alpha_k)$ ,  $\gamma_{k+1} = 1/(8L_{\rho_{k+1}}\alpha_{k+1})$ ,  $T_k = T_{k+1} = T_{k_0}$ ,  $\rho_k = 3s^{2/3}(k-k_0+7)^{4/3}/32$  or  $\rho_k = 3\sqrt{s}(k-k_0+7)/16$ . By these relations, (85), (92), and (93), one has that

$$\begin{aligned} \mathcal{L}_{k} - \mathcal{R}_{k+1} &\stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1)\gamma_{k} + (T_{k} - 1)\frac{\gamma_{k}p_{k}}{\alpha_{k}} - \frac{\gamma_{k+1}}{\alpha_{k+1}} + \gamma_{k+1} - (T_{k+1} - 1)\frac{\gamma_{k+1}p_{k+1}}{\alpha_{k+1}} \\ &= \frac{1}{8L_{\rho_{k}}\alpha_{k}^{2}} + (T_{k} - 1)\gamma_{k} + (T_{k} - 1)\frac{p_{k}}{8L_{\rho_{k}}\alpha_{k}^{2}} - \frac{1}{8L_{\rho_{k+1}}\alpha_{k+1}^{2}} + \gamma_{k+1} - (T_{k} - 1)\frac{p_{k}}{8L_{\rho_{k+1}}\alpha_{k+1}^{2}} \\ &= (T_{k} - 1)\Big(\gamma_{k} - \frac{p_{k}}{8L_{\rho_{k+1}}\alpha_{k+1}^{2}} + \frac{p_{k}}{8L_{\rho_{k}}\alpha_{k}^{2}}\Big) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}\alpha_{k+1}^{2}} + \frac{1}{8L_{\rho_{k}}\alpha_{k}^{2}} \\ &\geq (T_{k} - 1)\Big(\gamma_{k} - \frac{p_{k}}{8L_{\rho_{k}}}\Big(\frac{1}{\alpha_{k+1}^{2}} - \frac{1}{\alpha_{k}^{2}}\Big)\Big) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}}\Big(\frac{1}{\alpha_{k+1}^{2}} - \frac{1}{\alpha_{k}^{2}}\Big) (\text{due to } L_{\rho_{k}} \leq L_{\rho_{k+1}}) \end{aligned}$$

$$\begin{split} &= (T_k - 1) \left( \gamma_k - \frac{p_k}{8L_{\rho_k}} \frac{(k - k_0 + 8)^2 - (k - k_0 + 7)^2}{36} \right) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}} \frac{(k - k_0 + 8)^2 - (k - k_0 + 7)^2}{36} \\ &= (T_k - 1) \left( \gamma_k - \frac{p_k}{8L_{\rho_k}} \frac{2(k - k_0) + 15}{36} \right) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}} \frac{2(k - k_0) + 15}{36} \\ &\geq (T_k - 1) \left( \gamma_k - \frac{p_k}{8L_{\rho_k}} \frac{3(k - k_0 + 7)}{36} \right) + \gamma_{k+1} - \frac{1}{8L_{\rho_{k+1}}} \frac{2(k - k_0 + 8)}{36} \\ &= (T_k - 1) \left( \gamma_k - \frac{p_k}{16L_{\rho_k}\alpha_k} \right) + \gamma_{k+1} - \frac{1}{24L_{\rho_{k+1}}\alpha_{k+1}} \left( \text{due to } \alpha_k = 6/(k - k_0 + 7) \right) \\ &= (T_k - 1) \left( \gamma_k - \frac{p_k\gamma_k}{2} \right) + \gamma_{k+1} - \frac{\gamma_{k+1}}{3} \left( \text{due to } \gamma_k = 1/(8L_{\rho_k}\alpha_k) \right) \\ &= \frac{13}{14} (T_k - 1)\gamma_k + \frac{2\gamma_{k+1}}{3} = \frac{13}{14} (T_k - 1)\gamma_k + \frac{1}{12L_{\rho_{k+1}}\alpha_{k+1}} \left( \text{due to } \alpha_{k+1} \le \alpha_k \right) \\ &= \frac{13}{14} (T_k - 1)\gamma_k + \frac{1}{12L_{\rho_{k+1}}\alpha_k} \stackrel{(92)}{\geq} \frac{13}{14} (T_k - 1)\gamma_k + \frac{5}{72L_{\rho_k}\alpha_k} \left( \text{due to } \alpha_{k+1} \le \alpha_k \right) \\ &= \frac{13}{14} (T_k - 1)\gamma_k + \frac{5\gamma_k}{9} \ge \frac{8}{3(k - k_0 + 8)} \frac{k - k_0 + 7}{6} (T_k - 1)\gamma_k + \frac{8}{3(k - k_0 + 8)} \frac{k - k_0 + 7}{6} \gamma_k \\ &= \frac{8}{3(k - k_0 + 8)} \left( (T_k - 1) \frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} + \frac{\gamma_k}{\alpha_k} \right) \left( \text{due to } \alpha_k + p_k \le 1 \right) \\ &\geq \frac{2(\rho_{k+1} - \rho_k)}{\rho_{k+1}} \left( (T_k - 1) \frac{\gamma_k(\alpha_k + p_k)}{\alpha_k} + \frac{\gamma_k}{\alpha_k} \right) \stackrel{(85)}{=} \frac{2(\rho_{k+1} - \rho_k)}{\rho_{k+1}} \mathcal{L}_k, \end{split}$$

where the second equality follows from  $\gamma_k = 1/(8L_{\rho_k}\alpha_k)$ ,  $\gamma_{k+1} = 1/(8L_{\rho_{k+1}}\alpha_{k+1})$ , and  $T_k = T_{k+1}$ , and the fourth equality is due to  $\alpha_k = 6/(k - k_0 + 7)$  and  $\alpha_{k+1} = 6/(k - k_0 + 8)$ . Hence, (90) also holds for  $k > k_0$ .

We next derive upper bounds on  $F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k}$  and  $\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F^*_{\rho_k}]$ .

**Lemma 10.** Suppose that Assumptions 1 and 3 hold, and that  $\{\tilde{x}_k\}$  is generated by Algorithm 2 with  $\{q_i\}, \{\theta_t\}, \{\gamma_k\}, \{p_k\}, \{\alpha_k\}$  given in (27), and  $\{T_k\}, \{\rho_k\}$  given in (28) or (34). Let  $F_{\rho_k}^*, F_{\rho_k}, \mathcal{L}_k$  and  $\mathcal{R}_k$  be defined in (19) and (85) with such  $\{\rho_k\}, \{\gamma_k\}, \{p_k\}, \{\alpha_k\}, and \{T_k\}$ . Then for all  $k \geq 1$ , we have

$$F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*} \leq \mathcal{R}_{k}^{-1} \Big( \frac{1}{40L_{\rho_{1}}} \Big( D_{F} + \frac{\rho_{1}}{2} \| [c(x^{0})]_{+} \|^{2} \Big) + \frac{D_{\mathcal{X}}^{2}}{2} + \frac{\Lambda^{2}}{2} \sum_{i=1}^{k-1} \frac{\gamma_{i} T_{i}}{\rho_{i}} \\ + 4\Lambda^{2} \sum_{i=1}^{k-1} \frac{\mathcal{L}_{i}(\rho_{i+1} - \rho_{i})}{\rho_{i+1}^{2}} + 2L_{\nabla \bar{f}} D_{\mathcal{X}}^{2} \sum_{i=1}^{k-1} \gamma_{i} T_{i} \Big),$$
(94)

$$\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*] \le \mathcal{R}_k^{-1} \Big( \frac{1}{40L_{\rho_1}} \Big( D_F + \frac{\rho_1}{2} \| [c(x^0)]_+ \|^2 \Big) + \frac{D_{\mathcal{X}}^2}{2} + \frac{\Lambda^2}{2} \sum_{i=1}^{k-1} \frac{\gamma_i T_i}{\rho_i} + 4\Lambda^2 \sum_{i=1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} \Big),$$
(95)

where  $L_{\nabla \bar{f}}$ ,  $D_{\mathcal{X}}$ ,  $D_F$ , and  $\Lambda$  are given in (20) and Assumptions 1, respectively.

*Proof.* Notice from (27) and (68) that  $1/(8L_{\rho_k}\alpha_k) = \gamma_k \leq 1/(8L_{\nabla \bar{f}}\alpha_k)$ , which implies that  $L_{\rho_k}\alpha_k\gamma_k = 1/8$  and  $L_{\nabla \bar{f}}\alpha_k\gamma_k \leq 1/8$ . By these, one can easily verify that the assumption of Lemma 7 holds for

 $\{q_i\}, \{\alpha_k\}, \{\gamma_k\}$  and  $\{p_k\}$  given in (27). Let  $x^*$  be an arbitrary optimal solution of problem (4). It then follows from (82),  $F_{\rho_k}(x^*) = F^*$ , and (74) with  $x = x^*$  that

$$\frac{\gamma_k}{\alpha_k} (F_{\rho_k}(x_t) - F^*) \le \frac{\gamma_k}{\alpha_k} (1 - \alpha_k - p_k) (F_{\rho_k}(x_{t-1}) - F^*) + \frac{\gamma_k p_k}{\alpha_k} (F_{\rho_k}(\tilde{x}_k) - F^*) + \frac{1}{2} (\|z_{t-1} - x^*\|^2 - \|z_t - x^*\|^2) + 2\gamma_k L_{\nabla \bar{f}} D_{\mathcal{X}}^2.$$
(96)

Summing up (96) from t = 1 to  $T_k$ , and using the fact that  $x_0 = \tilde{x}_k$ ,  $z_0 = \tilde{z}_k$  and  $z_{T_k} = \tilde{z}_{k+1}$ , we have

$$\sum_{t=1}^{T_{k}} \frac{\gamma_{k}}{\alpha_{k}} (F_{\rho_{k}}(x_{t}) - F^{*}) \leq \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho_{k}}(x_{0}) - F^{*}) + \sum_{t=2}^{T_{k}} \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho_{k}}(x_{t-1}) - F^{*}) \\ + \frac{T_{k} \gamma_{k} p_{k}}{\alpha_{k}} (F_{\rho_{k}}(\tilde{x}_{k}) - F^{*}) + \frac{1}{2} (\|z_{0} - x^{*}\|^{2} - \|z_{T_{k}} - x^{*}\|^{2}) + 2\gamma_{k} T_{k} L_{\nabla \bar{f}} D_{\mathcal{X}}^{2} \\ = \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho_{k}}(\tilde{x}_{k}) - F^{*}) + \sum_{t=1}^{T_{k}-1} \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k} - p_{k}) (F_{\rho_{k}}(x_{t}) - F^{*}) \\ + \frac{T_{k} \gamma_{k} p_{k}}{\alpha_{k}} (F_{\rho_{k}}(\tilde{x}_{k}) - F^{*}) + \frac{1}{2} (\|\tilde{z}_{k} - x^{*}\|^{2} - \|\tilde{z}_{k+1} - x^{*}\|^{2}) + 2\gamma_{k} T_{k} L_{\nabla \bar{f}} D_{\mathcal{X}}^{2}.$$

$$(97)$$

By the definitions of  $\theta_t$  and  $\mathcal{L}_k$  in (27) and (85), one can observe that  $\mathcal{L}_k = \sum_{t=1}^{T_k} \theta_t$ . Using this, (97), the definition of  $\theta_t$  in (27), the convexity of  $F_{\rho_k}(\cdot)$ , and  $\tilde{x}_{k+1} = \sum_{t=1}^{T_k} (\theta_t x_t) / (\sum_{t=1}^{T_k} \theta_t)$ , we have

$$\mathcal{L}_{k}(F_{\rho_{k}}(\tilde{x}_{k+1}) - F^{*}) = \left(\sum_{t=1}^{T_{k}} \theta_{t}\right)(F_{\rho_{k}}(\tilde{x}_{k+1}) - F^{*}) \leq \sum_{t=1}^{T_{k}} \theta_{t}(F_{\rho_{k}}(x_{t}) - F^{*})$$

$$\stackrel{(27)}{=} \sum_{t=1}^{T_{k}} \frac{\gamma_{k}}{\alpha_{k}}(F_{\rho_{k}}(x_{t}) - F^{*}) - \sum_{t=1}^{T_{k}-1} \frac{\gamma_{k}}{\alpha_{k}}(1 - \alpha_{k} - p_{k})(F_{\rho_{k}}(x_{t}) - F^{*})$$

$$\stackrel{(97)}{\leq} \left(\frac{\gamma_{k}}{\alpha_{k}}(1 - \alpha_{k}) + (T_{k} - 1)\frac{\gamma_{k}p_{k}}{\alpha_{k}}\right)(F_{\rho_{k}}(\tilde{x}_{k}) - F^{*}) + \frac{1}{2}\left(\|\tilde{z}_{k} - x^{*}\|^{2} - \|\tilde{z}_{k+1} - x^{*}\|^{2}\right) + 2\gamma_{k}T_{k}L_{\nabla\bar{f}}D_{\mathcal{X}}^{2},$$

$$\stackrel{(85)}{=} \mathcal{R}_{k}(F_{\rho_{k}}(\tilde{x}_{k}) - F^{*}) + \frac{1}{2}\left(\|\tilde{z}_{k} - x^{*}\|^{2} - \|\tilde{z}_{k+1} - x^{*}\|^{2}\right) + 2\gamma_{k}T_{k}L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}.$$

$$(98)$$

Let  $x_k^*$  be an arbitrary optimal solution of (19). It then follows that  $F_{\rho_k}^* = F_{\rho_k}(x_k^*)$ . Notice from (19) and (85) that  $F_{\rho_{k+1}}^* \ge F_{\rho_k}^*$  and  $\mathcal{L}_k - \mathcal{R}_k = \gamma_k T_k \ge 0$ . Using these, (4), (19), (45), (46), (85), and (98), we obtain that for all  $k \ge 1$ ,

$$\frac{1}{2} \|\tilde{z}_{k} - x^{*}\|^{2} - \frac{1}{2} \|\tilde{z}_{k+1} - x^{*}\|^{2} + 2\gamma_{k}T_{k}L_{\nabla\bar{f}}D_{\mathcal{X}}^{2} \stackrel{(98)}{\geq} \mathcal{L}_{k}(F_{\rho_{k}}(\tilde{x}_{k+1}) - F^{*}) - \mathcal{R}_{k}(F_{\rho_{k}}(\tilde{x}_{k}) - F^{*}) \\
= \mathcal{L}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k+1}) - F_{\rho_{k+1}}^{*}\right) - \mathcal{R}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*}\right) + \mathcal{L}_{k}\left(F_{\rho_{k+1}}^{*} - F^{*}\right) - \mathcal{R}_{k}\left(F_{\rho_{k}}^{*} - F^{*}\right) \\
\geq \mathcal{L}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k+1}) - F_{\rho_{k+1}}^{*}\right) - \mathcal{R}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*}\right) + (\mathcal{L}_{k} - \mathcal{R}_{k})\left(F_{\rho_{k}}^{*} - F^{*}\right) \\
\stackrel{(4)(19)}{=} \mathcal{L}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k+1}) - F_{\rho_{k+1}}^{*}\right) - \mathcal{R}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*}\right) + \gamma_{k}T_{k}\left(F(x_{k}^{*}) - F^{*}\right) + \frac{\gamma_{k}T_{k}\rho_{k}}{2}\|[c(x_{k}^{*})]_{+}\|^{2} \\
\stackrel{(45)}{\geq} \mathcal{L}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k+1}) - F_{\rho_{k+1}}^{*}\right) - \mathcal{R}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*}\right) - \gamma_{k}T_{k}\Lambda\|[c(x_{k}^{*})]_{+}\| + \frac{\gamma_{k}T_{k}\rho_{k}}{2}\|[c(x_{k}^{*})]_{+}\|^{2} \\
\geq \mathcal{L}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k+1}) - F_{\rho_{k+1}}^{*}\right) - \mathcal{R}_{k}\left(F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*}\right) - \frac{\gamma_{k}T_{k}\Lambda^{2}}{2\rho_{k}} (due to Young's inequality)$$

where the last inequality is due to  $(a+b)^2 \leq 2(a^2+b^2)$  for all  $a, b \in \mathbb{R}$ . It then follows from (90), (99), and  $F_{\rho_{i+1}}(\tilde{x}_{i+1}) \geq F^*_{\rho_{i+1}}$  that for all  $i \geq 1$ ,

$$\mathcal{R}_{i+1}(F_{\rho_{i+1}}(\tilde{x}_{i+1}) - F_{\rho_{i+1}}^{*}) \stackrel{(90)}{\leq} \mathcal{L}_{i}\left(1 - \frac{2(\rho_{i+1} - \rho_{i})}{\rho_{i+1}}\right) (F_{\rho_{i+1}}(\tilde{x}_{i+1}) - F_{\rho_{i+1}}^{*}) \\
\stackrel{(99)}{\leq} \mathcal{R}_{i}(F_{\rho_{i}}(\tilde{x}_{i}) - F_{\rho_{i}}^{*}) + \frac{1}{2}\left(\|\tilde{z}_{i} - x^{*}\|^{2} - \|\tilde{z}_{i+1} - x^{*}\|^{2}\right) + 2\gamma_{i}T_{i}L_{\nabla\bar{f}}D_{\mathcal{X}}^{2} + \frac{\gamma_{i}T_{i}\Lambda^{2}}{2\rho_{i}} + \frac{4\mathcal{L}_{i}(\rho_{i+1} - \rho_{i})\Lambda^{2}}{\rho_{i+1}^{2}}.$$
(100)

By (27), (28), (34), (68), and (85), one can verify that  $\mathcal{R}_1 = 7/(288L_{\rho_1}) < 1/(40L_{\rho_1})$ . Summing up (100) from i = 1 to k - 1, and using Assumption 1(i),  $\tilde{x}_1 = \tilde{z}_1 = x^0$ , and  $\mathcal{R}_1 < 1/(40L_{\rho_1})$ , we have

$$\mathcal{R}_{k}(F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*}) \leq \frac{1}{40L_{\rho_{1}}}(F_{\rho_{1}}(x^{0}) - F_{\rho_{1}}^{*}) + \frac{1}{2}||x^{0} - x^{*}||^{2} + 2L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}\sum_{i=1}^{k-1}\gamma_{i}T_{i} + \frac{\Lambda^{2}}{2}\sum_{i=1}^{k-1}\frac{\gamma_{i}T_{i}}{\rho_{i}} + 4\Lambda^{2}\sum_{i=1}^{k-1}\frac{\mathcal{L}_{i}(\rho_{i+1} - \rho_{i})}{\rho_{i+1}^{2}}.$$

The conclusion (94) then follows from this and the fact that

$$F_{\rho_1}(x^0) - F_{\rho_1}^* \stackrel{(19)}{\leq} F(x^0) - F(x_1^*) + \frac{\rho_1}{2} \| [c(x^0)]_+ \|^2 \le D_F + \frac{\rho_1}{2} \| [c(x^0)]_+ \|^2,$$

where the last inequality is due to Assumption 1(i). In addition, the proof of (95) follows from (75) and similar arguments as above, and is thus omitted.  $\Box$ 

We are now ready to prove Theorem 4.

**Proof of Theorem 4.** Using (45) and (46) with  $\rho = \rho_k$  and  $x = \tilde{x}_k$ , we have

$$\|[c(\tilde{x}_k)]_+\| \stackrel{(46)}{\leq} \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2(F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*)}{\rho_k}},\tag{101}$$

$$\mathbb{E}[\|[c(\tilde{x}_k)]_+\|] \stackrel{(101)}{\leq} \frac{2\Lambda}{\rho_k} + \mathbb{E}\left[\sqrt{\frac{2(F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*)}{\rho_k}}\right] \leq \frac{2\Lambda}{\rho_k} + \sqrt{\frac{2\mathbb{E}\left[F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*\right]}{\rho_k}}.$$
 (102)

Also, notice from (27) that  $\alpha_k + p_k \leq 1$ . Moreover, as shown in the proof of Lemma 9, the sequence  $\{\rho_k\}$  defined in (28) or (34) is nondecreasing.

We first prove statement (i) of Theorem 4, where  $k_0$ ,  $\{T_k\}$ , and  $\{\rho_k\}$  are chosen as in (28). Notice from (27) and (28) that  $k_0 = \lfloor (4/3) \log_2 s \rfloor + 1$ ,  $\alpha_k = 6/7$ ,  $p_k = 1/7$ ,  $T_k = \lceil 2^{3(k-1)/4} \rceil$ ,  $\rho_k = 2^{k/2}$ , and  $\gamma_k = 1/(8(L_{\nabla \bar{f}} + 2^{k/2}L_{\nabla c^2}))$  for all  $1 \le k \le k_0$ . Using these relations and (85), we obtain that

$$\mathcal{L}_{k} \stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1) \frac{\gamma_{k}(\alpha_{k} + p_{k})}{\alpha_{k}} = \frac{T_{k}\gamma_{k}}{\alpha_{k}} = \frac{49 \times \lceil 2^{3(k-1)/4} \rceil}{288(L_{\nabla\bar{f}} + 2^{k/2}L_{\nabla c^{2}})} \le \frac{49 \times 2^{k/4}}{288L_{\nabla c^{2}}} \qquad \forall 1 \le k \le k_{0},$$
(103)

$$s/2 \le T_{k_0} = \lceil 2^{3(k_0-1)/4} \rceil = \lceil 2^{3\lfloor (4/3) \log_2 s \rfloor/4} \rceil \le s, \quad 2^{k_0/4} = 2^{(\lfloor (4/3) \log_2 s \rfloor + 1)/4} \le 2^{1/4} s^{1/3}.$$
(104)

By (27), (28), (85), (104), and  $\alpha_k + p_k \leq 1$ , one has that for all  $k > k_0$ ,

$$\mathcal{R}_{k} \stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k}) + (T_{k} - 1) \frac{\gamma_{k} p_{k}}{\alpha_{k}} \geq \frac{\gamma_{k} p_{k} T_{k}}{\alpha_{k}} \stackrel{(27)(28)}{=} \frac{T_{k_{0}} (k - k_{0} + 7)^{2}}{2016 (L_{\nabla \bar{f}} + (3s^{2/3} (k - k_{0} + 7)^{4/3}/32) L_{\nabla c^{2}})},$$

$$\stackrel{(104)}{\geq} \frac{s(k - k_{0} + 7)^{2}}{4032 (L_{\nabla \bar{f}} + (3s^{2/3} (k - k_{0} + 7)^{4/3}/32) L_{\nabla c^{2}})},$$

$$\mathcal{L}_{k} \stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1) \frac{\gamma_{k} (\alpha_{k} + p_{k})}{\alpha_{k}} \leq \frac{\gamma_{k} T_{k}}{\alpha_{k}} \stackrel{(27)(28)}{=} \frac{T_{k_{0}} (k - k_{0} + 7)^{2}}{288 (L_{\nabla \bar{f}} + (3s^{2/3} (k - k_{0} + 7)^{4/3}/32) L_{\nabla c^{2}})}$$

$$\stackrel{(104)}{\leq} \frac{s(k - k_{0} + 7)^{2}}{288 (L_{\nabla \bar{f}} + (3s^{2/3} (k - k_{0} + 7)^{4/3}/32) L_{\nabla c^{2}})} \leq \frac{s^{1/3} (k - k_{0} + 7)^{2/3}}{27L_{\nabla c^{2}}}.$$
(105)

In addition, observe that  $\lceil 2^{3(k-1)/4} \rceil \leq 2^{3k/4}$  for all  $k \geq 1$ . Using this relation,  $s \geq 1$ , (27), (28), and (104), we obtain that for all  $k \geq 1$ ,

$$\begin{split} \sum_{i=1}^{k-1} \gamma_i T_i &= \sum_{i=1}^{k_0} \gamma_i T_i + \sum_{i=k_0+1}^{k-1} \gamma_i T_i^{(27)(28)} \sum_{i=1}^{k_0} \frac{7T_i}{48(L_{\nabla\bar{f}} + \rho_i L_{\nabla c^2})} + \sum_{i=k_0+1}^{k-1} \frac{T_{k_0}}{8(L_{\nabla\bar{f}} + \rho_i L_{\nabla c^2})\alpha_i} \\ & \leq \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i} + \frac{s}{8L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i \alpha_i} \\ & (27)(28) \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{[2^{3(i-1)/4}]}{2^{i/2}} + \frac{2s^{1/3}}{9L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} (i - k_0 + 7)^{-1/3} \\ & \leq \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{3i/4}}{2^{i/2}} + \frac{2s^{1/3}}{9L_{\nabla c^2}} \int_{k_0+1}^{k} (\tau - k_0 + 6)^{-1/3} d\tau \\ & \leq \frac{7 \times 2^{k_0/4}}{48(1 - 2^{-1/4})L_{\nabla c^2}} + \frac{s^{1/3}(k - k_0 + 7)^{2/3}}{9L_{\nabla c^2}} \int_{k_0}^{(104)} \frac{11s^{1/3}}{10L_{\nabla c^2}} + \frac{s^{1/3}(k - k_0 + 7)^{2/3}}{3L_{\nabla c^2}}, \end{split}$$
(107)  
$$\sum_{i=1}^{k-1} \frac{\gamma_i T_i}{\rho_i} &= \sum_{i=1}^{k_0} \frac{\gamma_i T_i}{\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{\gamma_i T_i}{\rho_i} (27)(28)} \sum_{i=1}^{k_0} \frac{7T_i}{48(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{8(L_{\nabla \bar{f}} + \rho_i L_{\nabla c^2})\rho_i}{\rho_i \alpha_i} \\ & \begin{pmatrix} (104) \\ \leq \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i^2} + \frac{s}{8L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i^2 \alpha_i} \\ & \begin{pmatrix} (27)(28) \\ \sqrt{2} \\ \sqrt{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{1}{2^{3i/4}} + \frac{64s^{-1/3}}{27L_{\nabla c^2}} \int_{k_0+1}^{k-1} (i - k_0 + 7)^{-5/3} \\ & \leq \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{3i/4}}{2^i} + \frac{64s^{-1/3}}{27L_{\nabla c^2}} \int_{k_0+1}^{k-1} (i - k_0 + 6)^{-5/3} d\tau \end{split}$$

$$\leq \frac{7}{48(2^{1/4} - 1)L_{\nabla c^2}} + \frac{32}{27L_{\nabla c^2}} \leq \frac{2}{L_{\nabla c^2}}.$$
(108)

Observe from (28) that  $2^{k_0/2} = 2^{(\lfloor (4/3)\log_2 s \rfloor + 1)/2} \ge s^{2/3}$ . By this, (28), and the convexity of the functions  $2^{\tau/2}$  and  $\tau^{4/3}$ , one has

$$\rho_{i+1} - \rho_i = \begin{cases} 2^{(i+1)/2} - 2^{i/2} \le (2^{(i+1)/2} \log 2)/2 = (\rho_{i+1} \log 2)/2 & \text{if } 1 \le i < k_0, \\ 3s^{2/3}/2 - 2^{k_0/2} \le s^{2/3}/2 = \rho_{i+1}/3 \le (\rho_{i+1} \log 2)/2 & \text{if } i = k_0, \\ 3s^{2/3}((i-k_0+8)^{4/3} - (i-k_0+7)^{4/3})/32 \le s^{2/3}(i-k_0+8)^{1/3}/8 & \text{if } i > k_0. \end{cases}$$

$$(109)$$

Using (28), (103), (106), (109),  $s \ge 1$ , and  $\rho_{i+1} \ge \rho_i$ , we obtain that

$$\begin{split} &\sum_{i=1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2} = \sum_{i=1}^{k_0} \frac{\mathcal{L}_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2} + \sum_{i=k_0+1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1}-\rho_i)}{\rho_{i+1}^2} \\ &\stackrel{(109)}{\leq} \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_{i+1}} + \sum_{i=k_0+1}^{k-1} \frac{s^{2/3}\mathcal{L}_i(i-k_0+8)^{1/3}}{8\rho_{i+1}^2} \\ &\leq \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{s^{2/3}\mathcal{L}_i(i-k_0+8)^{1/3}}{8\rho_{i+1}^2} \quad (\text{due to } \rho_{i+1} \ge \rho_i) \\ &\leq \frac{49\log 2}{576L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{i/4}}{2^{i/2}} + \frac{128s^{-1/3}}{243L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{(i-k_0+7)^{2/3}(i-k_0+8)^{1/3}}{(i-k_0+8)^{8/3}} \quad (\text{due to } (28), (103), (106)) \\ &\leq \frac{49\log 2}{576(2^{1/4}-1)L_{\nabla c^2}} + \frac{128}{243L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} (i-k_0+8)^{-5/3} \le \frac{49\log 2}{576(2^{1/4}-1)L_{\nabla c^2}} + \frac{16}{81L_{\nabla c^2}} \le \frac{1}{L_{\nabla c^2}}. \end{split}$$

It follows from this, (33), (68), (94), (105), (107), (108),  $\rho_1 = \sqrt{2}$ , and  $s \ge 1$  that for all  $k > k_0$ ,

$$F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*} \stackrel{(94)}{\leq} \mathcal{R}_{k}^{-1} \Big( \frac{D_{F} + \rho_{1} \| [c(x^{0})]_{+} \|^{2}/2}{40L_{\rho_{1}}} + \frac{D_{\mathcal{X}}^{2}}{2} + \frac{\Lambda^{2}}{2} \sum_{i=1}^{k-1} \frac{\gamma_{i}T_{i}}{\rho_{i}} + 4\Lambda^{2} \sum_{i=1}^{k-1} \frac{\mathcal{L}_{i}(\rho_{i+1} - \rho_{i})}{\rho_{i+1}^{2}} \\ + 2L_{\nabla\bar{f}} D_{\mathcal{X}}^{2} \sum_{i=1}^{k-1} \gamma_{i}T_{i} \Big) \\ \leq \frac{4032L_{\nabla\bar{f}} + 378L_{\nabla c^{2}}s^{2/3}(k - k_{0} + 7)^{4/3}}{s(k - k_{0} + 7)^{2}} \Big( \frac{D_{F} + \sqrt{2} \| [c(x^{0})]_{+} \|^{2}/2}{40(L_{\nabla\bar{f}} + \sqrt{2}L_{\nabla c^{2}})} + \frac{D_{\mathcal{X}}^{2}}{2} + \frac{\Lambda^{2}}{L_{\nabla c^{2}}} \\ + \frac{4\Lambda^{2}}{L_{\nabla c^{2}}} + \frac{11L_{\nabla\bar{f}} D_{\mathcal{X}}^{2}s^{1/3}}{5L_{\nabla c^{2}}} + \frac{2L_{\nabla\bar{f}} D_{\mathcal{X}}^{2}s^{1/3}(k - k_{0} + 7)^{2/3}}{3L_{\nabla c^{2}}} \Big) \\ \stackrel{(33)}{\leq} \frac{C_{5}s^{-1} + C_{7}s^{-2/3}}{(k - k_{0} + 7)^{2}} + \frac{C_{6}s^{-1/3} + C_{8}}{(k - k_{0} + 7)^{2/3}} + \frac{C_{9}s^{-2/3}}{(k - k_{0} + 7)^{4/3}} + 252L_{\nabla\bar{f}} D_{\mathcal{X}}^{2} \\ \leq \frac{(C_{5} + C_{7})s^{-2/3}}{(k - k_{0} + 7)^{2}} + \frac{C_{6} + C_{8}}{(k - k_{0} + 7)^{2/3}} + \frac{C_{9}s^{-2/3}}{(k - k_{0} + 7)^{4/3}} + 252L_{\nabla\bar{f}} D_{\mathcal{X}}^{2},$$
(110)

where  $C_5$ ,  $C_6$ ,  $C_7$ ,  $C_8$ , and  $C_9$  are given in (33). Similarly, by (33) and (95), one has

$$\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*] \le \frac{C_5 s^{-1}}{(k - k_0 + 7)^2} + \frac{C_6 s^{-1/3}}{(k - k_0 + 7)^{2/3}}.$$
(111)

Further, using (95), (101), (102), and (110), we obtain that

$$\begin{split} \|[c(\tilde{x}_{k})]_{+}\| & \stackrel{(101)(110)}{\leq} \frac{2\Lambda}{\rho_{k}} + \sqrt{\frac{2(C_{5}+C_{7})s^{-2/3}}{(k-k_{0}+7)^{2}\rho_{k}}} + \frac{2(C_{6}+C_{8})}{(k-k_{0}+7)^{2/3}\rho_{k}} + \frac{2C_{9}s^{-2/3}}{(k-k_{0}+7)^{4/3}\rho_{k}} + \frac{504L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}}{\rho_{k}}}{\leq \frac{2\Lambda}{\rho_{k}} + \sqrt{\frac{2(C_{5}+C_{7})s^{-2/3}}{(k-k_{0}+7)^{2}\rho_{k}}} + \sqrt{\frac{2(C_{6}+C_{8})}{(k-k_{0}+7)^{2/3}\rho_{k}}} + \sqrt{\frac{2C_{9}s^{-2/3}}{(k-k_{0}+7)^{4/3}\rho_{k}}} + \sqrt{\frac{504L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}}{\rho_{k}}}\\ \mathbb{E}[\|[c(\tilde{x}_{k})]_{+}\|] \stackrel{(102)(111)}{\leq} \frac{2\Lambda}{\rho_{k}} + \sqrt{\frac{2C_{5}s^{-1}}{(k-k_{0}+7)^{2}\rho_{k}}} + \frac{2C_{6}s^{-1/3}}{(k-k_{0}+7)^{2/3}\rho_{k}}} \\ \leq \frac{2\Lambda}{\rho_{k}} + \sqrt{\frac{2C_{5}s^{-1}}{(k-k_{0}+7)^{2}\rho_{k}}} + \sqrt{\frac{2C_{6}s^{-1/3}}{(k-k_{0}+7)^{2/3}\rho_{k}}}, \end{split}$$

,

which, together with  $k \ge 2(k_0 - 7)$ ,  $\rho_k = 3s^{2/3}(k - k_0 + 7)^{4/3}/32$  and  $\mathbb{E}[F(\tilde{x}_k) - F^*] \ge -\Lambda \mathbb{E}[||[c(\tilde{x}_k)]_+||]$ due to (45), implies that (29), (30), and (32) hold. In addition, (31) follows from  $k \ge 2(k_0 - 7)$ , (45), and (111).

We next prove statement (ii) of Theorem 4, where  $k_0$ ,  $\{T_k\}$ , and  $\{\rho_k\}$  are chosen as in (34). Notice from (27) and (34) that  $k_0 = \lfloor \log_2 s \rfloor + 1$ ,  $\alpha_k = 6/7$ ,  $p_k = 1/7$ ,  $T_k = 2^{k-1}$ ,  $\rho_k = 2^{k/2}$ , and  $\gamma_k = 1/(8(L_{\nabla \bar{f}} + 2^{k/2}L_{\nabla c^2}))$  for all  $1 \le k \le k_0$ . Using these relations and (85), we obtain that

$$\mathcal{L}_{k} \stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1) \frac{\gamma_{k}(\alpha_{k} + p_{k})}{\alpha_{k}} = \frac{T_{k}\gamma_{k}}{\alpha_{k}} = \frac{49 \times 2^{k-1}}{288(L_{\nabla\bar{f}} + 2^{k/2}L_{\nabla c^{2}})} \le \frac{49 \times 2^{k/2}}{576L_{\nabla c^{2}}} \qquad \forall 1 \le k \le k_{0},$$
(112)

$$s/2 \le T_{k_0} = 2^{k_0 - 1} = 2^{\lfloor \log_2 s \rfloor} \le s, \quad 2^{k_0/2} = 2^{(\lfloor \log_2 s \rfloor + 1)/2} \le \sqrt{2s}.$$
 (113)

By (27), (34), (85), (113), and  $\alpha_k + p_k \leq 1$ , one has that for all  $k > k_0$ ,

$$\mathcal{R}_{k} \stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} (1 - \alpha_{k}) + (T_{k} - 1) \frac{\gamma_{k} p_{k}}{\alpha_{k}} \ge \frac{\gamma_{k} p_{k} T_{k}}{\alpha_{k}} \stackrel{(27)(34)}{=} \frac{T_{k_{0}} (k - k_{0} + 7)^{2}}{2016 (L_{\nabla \bar{f}} + (3\sqrt{s}(k - k_{0} + 7)/16)L_{\nabla c^{2}})},$$

$$\stackrel{(113)}{\ge} \frac{s(k - k_{0} + 7)^{2}}{4032 (L_{\nabla \bar{f}} + (3\sqrt{s}(k - k_{0} + 7)/16)L_{\nabla c^{2}})},$$

$$\mathcal{L}_{k} \stackrel{(85)}{=} \frac{\gamma_{k}}{\alpha_{k}} + (T_{k} - 1) \frac{\gamma_{k} (\alpha_{k} + p_{k})}{\alpha_{k}} \le \frac{\gamma_{k} T_{k}}{\alpha_{k}} \stackrel{(27)(34)}{\le} \frac{s(k - k_{0} + 7)^{2}}{288 (L_{\nabla \bar{f}} + (3\sqrt{s}(k - k_{0} + 7)/16)L_{\nabla c^{2}})},$$

$$\stackrel{(113)}{\le} \frac{\sqrt{s}(k - k_{0} + 7)}{54L_{\nabla c^{2}}}.$$

$$(115)$$

Using (27), (34), and (113), we obtain that for all  $k \ge 1$ ,

$$\sum_{i=1}^{k-1} \gamma_i T_i = \sum_{i=1}^{k_0} \gamma_i T_i + \sum_{i=k_0+1}^{k-1} \gamma_i T_i \overset{(27)(34)}{=} \sum_{i=1}^{k_0} \frac{7T_i}{48(L_{\nabla\bar{f}} + \rho_i L_{\nabla c^2})} + \sum_{i=k_0+1}^{k-1} \frac{T_{k_0}}{8(L_{\nabla\bar{f}} + \rho_i L_{\nabla c^2})\alpha_i} \\ \overset{(113)}{\leq} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i} + \frac{s}{8L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i \alpha_i} \overset{(27)(34)}{=} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{i-1}}{2^{i/2}} + \frac{\sqrt{s}}{9L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} 1 \\ \leq \frac{7 \times 2^{k_0/2}}{48(2 - \sqrt{2})L_{\nabla c^2}} + \frac{\sqrt{s}(k - k_0 - 1)}{9L_{\nabla c^2}} \overset{(113)}{\leq} \frac{7\sqrt{s}}{48(\sqrt{2} - 1)L_{\nabla c^2}} + \frac{\sqrt{s}(k - k_0 + 7)}{9L_{\nabla c^2}}, \tag{116}$$

$$\stackrel{(113)}{\leq} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{T_i}{\rho_i^2} + \frac{s}{8L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} \frac{1}{\rho_i^2 \alpha_i} \stackrel{(27)(34)}{=} \frac{7}{48L_{\nabla c^2}} \sum_{i=1}^{k_0} \frac{2^{i-1}}{2^i} + \frac{16}{27L_{\nabla c^2}} \sum_{i=k_0+1}^{k-1} (i-k_0+7)^{-1} \\ \leq \frac{7k_0}{96L_{\nabla c^2}} + \frac{16}{27L_{\nabla c^2}} \int_1^k \tau^{-1} d\tau \leq \frac{k_0}{12L_{\nabla c^2}} + \frac{3\log k}{5L_{\nabla c^2}}.$$

$$(117)$$

Observe from (34) that  $2^{k_0/2} = 2^{(\lfloor \log_2 s \rfloor + 1)/2} \ge \sqrt{s}$ . By this, (34), and the convexity of the function  $2^{\tau/2}$ , one has

$$\rho_{i+1} - \rho_i = \begin{cases} 2^{(i+1)/2} - 2^{i/2} \le (2^{(i+1)/2} \log 2)/2 \le (\rho_{i+1} \log 2)/2 & \text{if } 1 \le i < k_0, \\ 3\sqrt{s}/2 - 2^{k_0/2} \le \sqrt{s}/2 = \rho_{i+1}/3 \le (\rho_{i+1} \log 2)/2 & \text{if } i = k_0, \\ 3\sqrt{s}((i-k_0+8) - (i-k_0+7))/16 = 3\sqrt{s}/16 & \text{if } i > k_0. \end{cases}$$
(118)

Using (34), (112), (115), (118), and  $\rho_{i+1} \ge \rho_i$ , we obtain that

$$\begin{split} \sum_{i=1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} &= \sum_{i=1}^{k_0} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} + \sum_{i=k_0+1}^{k-1} \frac{\mathcal{L}_i(\rho_{i+1} - \rho_i)}{\rho_{i+1}^2} \stackrel{(118)}{\leq} \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_{i+1}} + \sum_{i=k_0+1}^{k-1} \frac{3\sqrt{s}\mathcal{L}_i}{16\rho_{i+1}^2} \\ &\leq \sum_{i=1}^{k_0} \frac{\mathcal{L}_i \log 2}{2\rho_i} + \sum_{i=k_0+1}^{k-1} \frac{3\sqrt{s}\mathcal{L}_i}{16\rho_{i+1}^2} \quad (\text{due to } \rho_{i+1} \ge \rho_i) \\ &\leq \frac{49 \log 2}{1152L_{\nabla C^2}} \sum_{i=1}^{k_0} 1 + \frac{8}{81L_{\nabla C^2}} \sum_{i=k_0+1}^{k-1} \frac{i - k_0 + 7}{(i - k_0 + 8)^2} \quad (\text{due to } (34), (112), (115)) \\ &\leq \frac{49k_0 \log 2}{1152L_{\nabla C^2}} + \frac{8}{81L_{\nabla C^2}} \sum_{i=k_0+1}^{k-1} (i - k_0 + 7)^{-1} \leq \frac{k_0}{24L_{\nabla C^2}} + \frac{\log k}{10L_{\nabla C^2}}. \end{split}$$

It follows from this, (39), (68), (94), (114), (116), (117),  $\rho_1 = \sqrt{2}$ , and  $s \ge 1$  that for all  $k > k_0$ ,

$$\begin{split} F_{\rho_{k}}(\tilde{x}_{k}) - F_{\rho_{k}}^{*} \stackrel{(94)}{\leq} \mathcal{R}_{k}^{-1} \Big( \frac{D_{F} + \rho_{1} \| [c(x^{0})]_{+} \|^{2}/2}{40L_{\rho_{1}}} + \frac{D_{\chi}^{2}}{2} + \frac{\Lambda^{2}}{2} \sum_{i=1}^{k-1} \frac{\gamma_{i}T_{i}}{\rho_{i}} + 4\Lambda^{2} \sum_{i=1}^{k-1} \frac{\mathcal{L}_{i}(\rho_{i+1} - \rho_{i})}{\rho_{i+1}^{2}} \\ &+ 2L_{\nabla \bar{f}} D_{\chi}^{2} \sum_{i=1}^{k-1} \gamma_{i} T_{i} \Big) \\ \leq \frac{4032L_{\nabla \bar{f}} + 756\sqrt{s}(k - k_{0} + 7)L_{\nabla c^{2}}}{s(k - k_{0} + 7)^{2}} \Big( \frac{D_{F} + \sqrt{2} \| [c(x^{0})]_{+} \|^{2}/2}{40(L_{\nabla \bar{f}} + \sqrt{2}L_{\nabla c^{2}})} + \frac{D_{\chi}^{2}}{2} + \frac{\Lambda^{2}k_{0}}{24L_{\nabla c^{2}}} \\ &+ \frac{3\Lambda^{2}\log k}{10L_{\nabla c^{2}}} + \frac{\Lambda^{2}k_{0}}{6L_{\nabla c^{2}}} + \frac{2\Lambda^{2}\log k}{5L_{\nabla c^{2}}} + \frac{7L_{\nabla \bar{f}} D_{\chi}^{2}\sqrt{s}}{24(\sqrt{2} - 1)L_{\nabla c^{2}}} + \frac{2L_{\nabla \bar{f}} D_{\chi}^{2}\sqrt{s}(k - k_{0} + 7)}{9L_{\nabla c^{2}}} \Big) \\ \stackrel{(39)}{\leq} \frac{C_{10}s^{-1} + C_{13}s^{-1/2}}{(k - k_{0} + 7)^{2}} + \frac{C_{11}s^{-1/2} + C_{14} + C_{15}s^{-1/2}}{k - k_{0} + 7} + \frac{C_{12}s^{-1}\log k}{(k - k_{0} + 7)^{2}} + \frac{523\Lambda^{2}s^{-1/2}\log k}{k - k_{0} + 7} \\ &+ 168L_{\nabla \bar{f}} D_{\chi}^{2} \\ \leq \frac{(C_{10} + C_{13})s^{-1/2}}{(k - k_{0} + 7)^{2}} + \frac{C_{11} + C_{14} + C_{15}}{k - k_{0} + 7} + \frac{C_{12}s^{-1}\log k}{(k - k_{0} + 7)^{2}} + \frac{523\Lambda^{2}s^{-1/2}\log k}{k - k_{0} + 7} \\ &+ 168L_{\nabla \bar{f}} D_{\chi}^{2}, \end{split}$$
(119)

where  $C_{10}$ ,  $C_{11}$ ,  $C_{12}$ ,  $C_{13}$ ,  $C_{14}$ , and  $C_{15}$  are given in (39). Similarly, by (39) and (95), one has

$$\mathbb{E}[F_{\rho_k}(\tilde{x}_k) - F_{\rho_k}^*] \le \frac{C_{10}s^{-1}}{(k - k_0 + 7)^2} + \frac{C_{11}s^{-1/2}}{k - k_0 + 7} + \frac{C_{12}s^{-1}\log k}{(k - k_0 + 7)^2} + \frac{523\Lambda^2 s^{-1/2}\log k}{k - k_0 + 7}.$$
 (120)

Using (101), (102), (119), and (120), we have

$$\begin{split} \|[c(\tilde{x}_{k})]_{+}\| &\stackrel{(101)(119)}{\leq} \frac{2\Lambda}{\rho_{k}} + \sqrt{\frac{2(C_{10} + C_{13})s^{-1/2}}{(k - k_{0} + 7)^{2}\rho_{k}}} + \sqrt{\frac{2(C_{11} + C_{14} + C_{15})}{(k - k_{0} + 7)\rho_{k}}} \\ &+ \sqrt{\frac{2C_{12}s^{-1}\log k}{(k - k_{0} + 7)^{2}\rho_{k}}} + \sqrt{\frac{1046\Lambda^{2}s^{-1/2}\log k}{(k - k_{0} + 7)\rho_{k}}} + \sqrt{\frac{336L_{\nabla\bar{f}}D_{\mathcal{X}}^{2}}{\rho_{k}}}, \\ \mathbb{E}[\|[c(\tilde{x}_{k})]_{+}\|] &\stackrel{(102)(120)}{\leq} \frac{2\Lambda}{\rho_{k}} + \sqrt{\frac{2C_{10}s^{-1}}{(k - k_{0} + 7)^{2}\rho_{k}}} + \sqrt{\frac{2C_{11}s^{-1/2}}{(k - k_{0} + 7)\rho_{k}}} + \sqrt{\frac{2C_{12}s^{-1}\log k}{(k - k_{0} + 7)^{2}\rho_{k}}} \\ &+ \sqrt{\frac{1046\Lambda^{2}s^{-1/2}\log k}{(k - k_{0} + 7)\rho_{k}}}, \end{split}$$

which, together with (45),  $\rho_k = 3\sqrt{s(k-k_0+7)}/16$ , and the relation  $k-k_0+7 \ge k/2$  for all  $k \ge \max\{k_0+1, 2(k_0-7)\}$ , imply that (35), (36), and (38) hold. In addition, (37) follows from  $k \ge 2(k_0-7)$ , (45), and (120).

# 7 Concluding remarks

In this paper, we proposed first-order methods for computing an  $\epsilon$ -surely feasible stochastic optimal ( $\epsilon$ -SFSO) solution for both stochastic and finite-sum convex optimization problems with deterministic constraints, and established first-order oracle (FO) complexity results for these methods. As a byproduct, we also derived FO complexity results for the sample average approximation (SAA) method in computing an  $\epsilon$ -SFSO solution of the stochastic optimization problem, using the proposed stochastic first-order method (Algorithm 2) to solve the sample average problem.

Although we focused on problem (1) with convex inequality constraints, our algorithms and results can be directly extended to problems that include both affine equality constraints and convex inequality constraints. Furthermore, they can also be extended to problems with convex conic constraints of the form  $c(x) \in -\mathcal{K}$ , where  $\mathcal{K}$  is a closed convex cone whose projection operator can be evaluated exactly, and c is both Lipschitz smooth and  $\mathcal{K}$ -convex; that is,  $\alpha c(x) + (1 - \alpha)c(y) - c(\alpha x + (1 - \alpha)y) \in \mathcal{K}$  for all  $x, y \in \mathbb{R}^n$  and  $\alpha \in [0, 1]$ .

An interesting direction for future research is to develop a stochastic first-order method for solving finite-sum convex optimization problems with deterministic constraints that achieves an FO complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-1})$  for computing an  $\epsilon$ -SFSO solution. This would significantly improve upon the current complexity of  $\mathcal{O}(s \log s + \sqrt{s}\epsilon^{-3/2})$  achieved by our proposed method, and would match the optimal complexity known for computing an  $\epsilon$ -expectedly feasible stochastic optimal solution.

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